A Functional Central Limit Theorem and its Bootstrap Analogue for Locally Stationary Processes with Application to Independence Testing

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Donald Duck: Hello!
The True Spirit of Adventure: Hello Donald.
Donald Duck: That's me! Where am I?
The True Spirit of Adventure: Mathmagic Land.
Donald Duck: Mathmagic Land? Never heard of it.
The True Spirit of Adventure: It's the land of great adventure.
Donald Duck: Well, who are you?
The True Spirit of Adventure: I'm a spirit. The true spirit of adventure.
Donald Duck: That's for me! What's next?
The True Spirit of Adventure: A journey through the wonderland of mathematics.

Donald in Mathmagic Land (1959)

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Summary

The outline of this thesis orientates itself on its title. In the second chapter, right after the introduction, we aim to establish a functional central limit theorem (FCLT) for a special class of processes based on the concept of local stationarity as the first step. This requires the determination of a framework we are going to adjust in case of necessity throughout the thesis. Proceeding from the idea of generalizing the findings in Jentsch et al. (2020), we identify two main frameworks which differ in terms of the boundedness of the used function. This division runs through the whole theoretical part in a more or less prominent way. In Chapter 2, we also present the results leading eventually to the statement of a FCLT. These results include findings addressing the covariance, a central limit theorem and a tightness result.

The next chapter covers the second part of the title as it transfers the results of the previous chapter to the bootstrap world. After a short overview of the concept of local block bootstrap, we adapt the algorithm of Dowla et al. (2013) to our framework. Then, we establish the bootstrap counterparts of the findings made in Chapter 2. In doing so, auxiliary results fitted to the bootstrap world are presented as well. Besides, during this chapter the segmentation regarding the boundedness of the used function becomes more evident.

Subsequently, we conduct a first simulation study in Chapter 4. In this context, we pick up on the empirical characteristic function the results in Jentsch et al. (2020) are based upon. Therefore, we begin this chapter with some tailor-made results for this special case. Similar to the simulations in the aforesaid paper, we choose a representative of the class of α-stable distributions to serve as the underlying distribution for our simulations. After briefly addressing the benefits of this particular class, the fineness of our bootstrap procedure is shown for different parameter choices along with our interpretations.

The remaining three chapters are dedicated to the last part of the title as they focus on a testing procedure for independence resting on a weighted distance composed of characteristic functions (CFs) and its empirical version. The first of them, Chapter 5, presents the basic idea of this concept, which is inspired by the distance covariance defined by Székely et al. (2007). Besides, as distance covariance is also used in Jentsch et al. (2020), we show the differences to our version with respect to both papers. In order to compile a testing procedure at the end, we provide the results needed for this purpose. Eventually, said procedure is presented with the notion of the beneficial effects of a bootstrap analogue, which leads to the following chapter.

In Chapter 6, we proceed in the same manner as in Chapter 3 to establish the bootstrap versions of the findings presented in the previous one. However, prior to that we transfer the concept of empirical weighted CF-distance to the bootstrap world.

Finally, Chapter 7 contains a simulation study using our previously presented testing procedure to detect dependence. After an adjustment of the simulation setup used in Chapter 4, several tables show the non-rejections rates using both underlying independent and dependent innovation series. Our interpretation of these results closes the main part of this thesis.

In the first part of the appendix, the proofs belonging to the results established in the main part can be found along with some auxiliary results. Moreover, Chapter B, contains the codes for the simulations as well as some runtime improving calculations.

Zusammenfassung

Der Aufbau dieser Dissertation wird durch ihren Titel bestimmt. Nach der Einleitung verfolgt das zweite Kapitel, gemäß dem ersten Teil des Titels, die Erarbeitung eines funktionalen zentralen Grenzwertsatzes (FCLT) für eine spezielle Klasse von Prozessen basierend auf dem Konzept der lokalen Stationarität. Dazu werden zuerst die Rahmenbedingungen festgelegt, die, je nach Notwendigkeit, im Laufe der Arbeit angepasst werden. Ausgehend von der Idee, die Resultate aus Jentsch et al. (2020) zu verallgemeinern, werden zwei verschiedene Hauptszenarien identifiziert, die sich in der Beschränktheit der verwendeten Funktion unterscheiden. Die Trennung dieser beiden Szenarien zieht sich danach durch den gesamten theoretischen Teil in mehr oder weniger prominenter Weise. In Kapitel 2 werden ebenfalls die Resultate präsentiert, die schlussendlich zur Formulierung des FCLTs führen. Darin eingeschlossen sind sowohl Resultate über die Kovarianz als auch ein zentraler Grenzwertsatz sowie ein Straffheitsresultat.

Das nächste Kapitel deckt den zweiten Teil des Titels ab, da es die Resultate aus dem vorherigen Kapitel in die Bootstrap-Welt überträgt. Nach einem kurzen Überblick hinsichtlich des Konzepts des lokalen Block-Bootstrap wird der Algorithmus von Dowla et al. (2013) den nun vorherrschenden Gegebenheiten angepasst. Daraufhin werden die Bootstrap-Gegenstücke zu den Resultaten aus Kapitel 2 erarbeitet. Dabei werden ebenfalls Hilfsresultate, die speziell auf die Bootstrap-Welt zugeschnitten sind, vorgestellt. Außerdem wird die Trennung der Resultate bezüglich der Beschränktheit der verwendeten Funktion offensichtlicher.

Anschließend wird in Kapitel 4 eine erste Simulationsstudie durchgeführt. In diesem Zusammenhang werden die empirischen charakteristischen Funktionen, auf denen die Resultate in Jentsch et al. (2020) basieren, aufgegriffen. Deswegen beginnt das Kapitel mit einigen Resultaten, die genau für diesen Spezialfall gemacht sind. Ähnlich zu den Simulationen in der zuvor erwähnten Arbeit fungiert auch hier ein Vertreter der Klasse der α-stabilen Verteilungen als zugrundeliegende Verteilung für die Simulationen. Dann, nach einem kurzen Abriss über die Vorteile dieser Verteilungsklasse, werden die Simulationsergebnisse für verschiedene Parameterwahlen nebst zugehörigen Interpretationen präsentiert.

Die verbleibenden drei Kapitel sind dem letzten Teil des Titels dieser Arbeit gewidmet. Mit anderen Worten, sie legen ihr Augenmerk auf ein Testverfahren auf Unabhängigkeit, dass auf der später definierten *weighted characteristic function (CF) distance* und deren empirischer Version fußt. Das erste der drei, nämlich Kapitel 5, präsentiert die Grundidee dieses Konzepts, welches in Anlehnung an die *distance covariance*, wie sie bei Székely et

al. (2007) definiert wird, entstanden ist. Da auch in Jentsch et al. (2020) distance covariance ihre Anwendung findet, werden außerdem die Unterschiede zu beiden Arbeiten thematisiert. Um schlussendlich ein Testverfahren vorstellen zu können, werden im Anschluss die dazu benötigten Resultate dargelegt. Am Ende des Kapitels kann dann besagtes Testverfahren präsentiert werden. Dabei wird auf die implementarischen Vorteile eines Bootstrap-Analogons aufmerksam gemacht, was direkt zum nächsten Kapitel überleitet.

In Kapitel 6 wird in der gleichen Weise fortgefahren wie zuvor schon in Kapitel 3, um die Bootstrap-Versionen der Resultate aus dem vorherigen Kapitel zu erarbeiten. Doch zuvor wird das Konzept der empirischen weighted CF-distance in die Bootstrap-Welt übertragen.

Abschließend enthält Kapitel 7 eine Simulationsstudie, die das zuvor vorgestellte Testverfahren nutzt, um Abhängigkeit zu erkennen. Nach Anpassungen des Simulationssetups aus Kapitel 4 zeigen verschiedene Tabellen die Nichtverwerfungsraten auf Grundlage von sowohl unabhängigen wie abhängigen Innovationen. Die zugehörigen Interpretationen der Ergebnisse beenden letztendlich den Hauptteil dieser Dissertation.

Im ersten Teil des Anhangs befinden sich die zu den im Hauptteil vorgestellen Resultaten gehörenden Beweise. Außerdem beinhaltet Kapitel A auch einige zusätzliche Hilfsresultate. Des Weiteren umfasst Kapitel B die Codes für die Simulationsstudien zusammen mit laufzeitverbessernden Rechnungen.

1 Introduction

Normally, ‘Do you believe in coincidence?’ is a question with only two possible answers, either ‘yes’ or ‘no’. The former implies there is no predetermined destiny, and unforeseen events can take place anytime. The latter, in turn, signifies a predefined sequence of events will certainly take place because it is written in a greater plan. But there is a third way of dealing with happenstance, namely the way examined in stochastic theory. There, the concept of coincidence is called *randomness*. In the stochastic world, we believe that there are some possible scenarios for, let’s say, a future event, but the probability of each scenario is predetermined. Consequently, in this world the answer to the aforementioned question is not as clear as generally assumed. But randomness and uncertainty go hand in hand. More than often, the probabilities are fixed but unknown at the same time. This leads to the need of procedures to manage the uncertainty for instance by estimating quantities or parameters based on past events. The question attached to these procedures is how valuable the results are.

As the mention of future and past events suggests, it is quite common to record and then model certain things over time, for example prices, temperatures or heartbeats. These series of data points, whose indices are time-ordered, are called *time series*.

At this point, we arrive at the scene where this thesis has its entrance. In the following chapters, we go on a travel through different worlds, make partly long-winded detours to pass scenic outposts and are rewarded at the end with a presentation of our souvenirs from the ride in application. However, to be prepared for this travel, we need to study a short guidebook addressing stochastic processes and stationarity first. Afterwards, equipped with this foreknowledge we have a closer look at our itinerary.

The following definitions are based on the ones to be found in Brockwell and Davis (1991) and Kreiß and Neuhaus (2006).

As already mentioned, time-based records of events are called time-series. If we want to model these events, we use random objects joint with a probability space $(\Omega, \mathcal{A}, P)$, where Ω is a sample space, $\mathcal{A}$ a σ-algebra and P a probability measure. Nevertheless, this concept is not restricted to time. A general definition reads as follows:

Definition 1.1 (Stochastic Process).
Let $(\Omega, \mathcal{A}, P)$ be a probability space. Then, a thereon defined family of random objects $(X_t)_{t\in T}$, $T \in \mathbb{Z}$, is called *stochastic process*.

Stochastic processes can be 1- or d-dimensional with $d \in \mathbb{N}_{>1}$. In this thesis, we will focus on the more general version of a vector-sized stochastic process, which includes the 1-dimensional variety automatically. To indicate the potential multi-dimensionality, we will underline the random objects. However, in this short overview we stick to the notation from above and, thus, to the 1-dimensional case. Nevertheless, the definitions can be transferred straightforwardly to the higher dimensional case. Now we move on with the next definition:

Definition 1.2 (Sample Path).
For fixed $\omega \in \Omega$ the functions $X_{\cdot}(\omega)$ on $T \in \mathbb{Z}$ are called the process' *realizations* or *sample-paths.*

A helpful characteristic of some stochastic processes, which implies a certain regularity, is stationarity. Because of this regularity, more extensive results can be shown compared to a non-stationary setting. By going into defining details, we have to distinguish between two different types of stationarity. The first reads as follows:

Definition 1.3 (Strict Stationarity).
A stochastic process $(X_t)_{t\in\mathbb{Z}}$ is said to be *strictly stationary* if it holds

$$\mathcal{L}\left(X_{t_1}, \ldots, X_{t_r}\right) = \mathcal{L}\left(X_{t_1+h}, \ldots, X_{t_r+h}\right)$$

with $h \in \mathbb{N}$ for all $t \in \mathbb{Z}$ and $r \in \mathbb{N}$.

The second type is weaker but not necessarily implied by the already defined one.

Definition 1.4 (Stationarity).
A stochastic process $(X_t)_{t\in\mathbb{Z}}$ is called *stationary* if the following conditions are satisfied:

(i) For all $t \in \mathbb{Z}$ it holds $E\left|X_t\right|^2 < \infty$.

(ii) The first moments are the same for all $t \in \mathbb{Z}$.

(iii) The covariance $\operatorname{Cov}\left(X_t, X_{t+h}\right)$ is only depending on $h \in \mathbb{Z}$.

However, comparing these two definitions above we notice that the strict stationarity combined with finite second absolute moments does allow for the fulfilment of the stationary requirements belonging to the weaker type.

Now we have a look at some particular types of processes, which will play an important role throughout the thesis. The first one is the so-called *white noise.*

Definition 1.5 (White Noise).
A stochastic process $(\varepsilon_t)_{t\in\mathbb{Z}}$ is called *white noise* if it has zero mean and fulfils

$$\operatorname{Cov}\left(\varepsilon_s, \varepsilon_t\right) = \begin{cases} \sigma_\varepsilon^2, & s = t, \\ 0, & s \neq t, \end{cases}$$

for some $\sigma_\varepsilon^2 \in (0, \infty)$.

The next two definitions address types of stochastic processes which are built with use of the above-defined white noise.

Definition 1.6 (Moving Average (MA) Process).
Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise. For real- or complex-valued coefficients $\mathsf{b}_1, \ldots, \mathsf{b}_q$ for $q \in \mathbb{N}$ with $\mathsf{b}_q \neq 0$,

$$X_t := \varepsilon_t + \mathsf{b}_1\,\varepsilon_{t-1} + \cdots + \mathsf{b}_q\,\varepsilon_{t-q}$$

for all $t \in \mathbb{Z}$ defines a stationary process, which is called a *moving average process of order* q or *MA(q)-process.*

At this point, we refer to Kreiß and Neuhaus (2006) for a verification of the claimed stationarity. The following definition is less general than the previous one because we restrict ourselves to the order 1.

Definition 1.7 (First Order Autoregressive Process).
A stochastic process $(X_t)_{t\in\mathbb{Z}}$ is called a *first order autoregressive process* or, in short, *AR(1)-process* if there exist a parameter $\mathsf{a} \in \mathbb{C}$ and a white noise $(\varepsilon_t)_{t\in\mathbb{Z}}$ for which it holds

$$X_t + \mathsf{a}\,X_{t-1} = \varepsilon_t \tag{1.1}$$

for all $t \in \mathbb{Z}$.

In contrast to the MA(q)-process, there is no guaranteed stationary solution for a given white noise $(\varepsilon_t)_{t\in\mathbb{Z}}$ and parameter $\mathsf{a} \in \mathbb{C}$ to satisfy equation (1.1). For details, we reference, again, Kreiß and Neuhaus (2006). Nevertheless, if such a stationary solution exists, it has the form of a MA(∞)-process, whose definition reads as follows:

Definition 1.8 (MA(∞)-Process).
Let $(\varepsilon)_{t\in\mathbb{Z}}$ be a white noise. A stochastic process $(X_t)_{t\in\mathbb{Z}}$ is said be a *MA(∞)-process* if there exist coefficients $(\psi_j)_{j\in\mathbb{Z}}$ satisfying $\sum_{j\in\mathbb{Z}} |\psi_j| < \infty$ such that X_t can be written as

$$X_t = \sum_{j\in\mathbb{Z}} \psi_j\,\varepsilon_{t-j} \tag{1.2}$$

for every $t \in \mathbb{Z}$.

Now that we have familiarity with the basics, we focus on the stages of our journey ahead. This thesis is divided into two parts, a main one and an appendix. In the main part, we present the surroundings of our soon-to-be-established results and the findings themselves. Moreover, the simulation studies find their place there. The appendix, on the other hand, has a backing function. First, the proofs and auxiliary results can be found there. In addition to that, the codes for the different simulations and supplementary computations are shifted towards this part.

To describe the two parts more in detail, we travel through two worlds in them main part: the so-called *real world* in Chapter 2 and the *bootstrap world* in the chapter thereafter. In the former, we establish a functional central limit theorem (FCLT) for locally stationary processes, whereas in the latter we present bootstrap versions of these findings. To join a practical part, in Chapter 4 we apply the aforementioned bootstrap results to α-stable distributions and conduct a simulation study therewith. Chapters 5 to 7 are dedicated to a second application, which consists of proposing a test for independence using a weighted distance based on characteristic functions and its empirical counterpart. In the first of these three chapters, we explain the idea and present the corresponding results. Chapter 6, again, shows the bootstrap analogues. Lastly, we close with a simulation-focussed chapter to apply our theoretical findings in practice.

Before we begin our journey, we turn our attention to some notational aspects. To diminish repeated enumeration, we use the following notation throughout this thesis without further reminder:

$$T, d \in \mathbb{N}, \quad \delta \in (0,1) \quad \text{and} \quad S \in \mathbb{R}_+.$$

Here, $\mathbb{N}$ is defined without 0. Otherwise, we would write $\mathbb{N}_0$. Besides, $\mathbb{R}_+$ denotes the positive, that is greater than 0, real numbers. Moreover, every time a variable is defined depending on T this definition holds for every T.

Furthermore, *i.i.d.* stands for *independent and identically distributed.*

In this thesis, we work with different types of constants. First, there will be specified ones, either per definition or originating from the proofs. On the other hand, we will use auxiliary constants, which we do not specify further. The last-mentioned will be denoted with C or C_j for $j \in \mathbb{N}$. Those will always be positive and finite. Besides, they may vary from line to line.

As already mentioned apropos of random objects, vector-valued quantities will be marked by underlining. In this context, $\underline{0}$ signifies the zero vector. Besides, $\underline{\mathsf{x}}'$ labels the transpose of a vector $\underline{\mathsf{x}}$. The transpose of a matrix is indicated in the same manner. In this context, ∇ denotes the gradient and is a row vector.

Next, we introduce some norms:

- First, we stick to the usual definition of the p-norm with $1 \leq p < \infty$ for d-dimensional vectors $\underline{\mathsf{v}} = (\mathsf{v}_1, \ldots, \mathsf{v}_n)'$, that is

$$|\underline{\mathsf{v}}|_p := \left(\sum_{j=1}^{d} |\mathsf{v}_j|^p \right)^{1/p}.$$

- Moreover, we denote the *max column sum matrix norm* by $|\cdot|_1$. More precisely, for some matrix $\mathsf{M} = (\mathsf{m}^{(i,j)})$ of dimension $(d \times r)$ with $n \in \mathbb{N}$ as well we have

$$|\mathsf{M}|_1 := \max_{1\le j\le r} \sum_{i=1}^{d} |\mathsf{m}^{(i,j)}|.$$

This norm is submultiplicative. Additionally, for $r = 1$ we obtain the 1-norm for vectors of dimension d.

- Furthermore, we signify the $\mathcal{L}^p$-norm with $1 \le p < \infty$ for d-dimensional random vectors $\underline{X}$ as usual by $\|\underline{X}\|_p := \left(E\,|\underline{X}|_p\right)^{1/p}$.
- In addition to that, we transfer said notation to the bootstrap world by defining $\|\underline{X}^\star\|_{p,\star} := \left(E^\star\,|\underline{X}^\star|_p\right)^{1/p}$ to be the bootstrap $\mathcal{L}^p$-norm of $\underline{X}^\star$.
- For a function $\mathsf{f}\colon \mathsf{T} \to \mathbb{R}$ with $\mathsf{T} \subset \mathbb{R}^d$, the infinity norm is defined as

$$\sup_{\underline{\mathsf{x}}\in\mathsf{T}} \left|\mathsf{f}\left(\underline{\mathsf{x}}\right)\right|.$$

- Besides, the Lipschitz seminorm $|\cdot|_{\mathrm{Lip}}$ of a function f as above stands for

$$|\mathsf{f}|_{\mathrm{Lip}} := \sup_{\substack{\underline{\mathsf{x}},\underline{\mathsf{y}}\in\mathsf{T}\\ \underline{\mathsf{x}}\ne\underline{\mathsf{y}}}} \frac{\left|f\left(\underline{\mathsf{x}}\right) - f\left(\underline{\mathsf{y}}\right)\right|}{\left|\underline{\mathsf{x}} - \underline{\mathsf{y}}\right|_1}.$$

Turning away from the norms, we move on to floor and ceiling functions. For $\mathsf{x} \in \mathbb{R}$, $\lfloor\mathsf{x}\rfloor$ denotes the largest integer smaller than or equal to x. In the same way, $\lceil\mathsf{x}\rceil$ stands for the smallest integer greater than or equal to x. In addition to that, $\langle\cdot,\cdot\rangle$ signifies the Euclidean inner product. For $\mathsf{z} \in \mathbb{C}$, we denote the real and imaginary part of z by $\Re\,\mathsf{z}$ and $\Im\,\mathsf{z}$, respectively. Furthermore, the function Φ, as usual, labels the distribution function of the standard Gaussian distribution.

To finish this notation part, we address different forms of convergence. On the one hand, $\xrightarrow{d}$ denotes convergence in distribution. On the other, $\xrightarrow{P}$ stands for stochastic convergence in the real world. The latter convergence has an analogue using $P^\star$ in place of P in the bootstrap world. Moreover, we use the big and small O notation in probability. To this end, consider a series of random variables or vectors $(\underline{X}_T)_{T\in\mathbb{N}}$ and a corresponding set of positive constants $(\mathsf{a}_T)_{T\in\mathbb{N}}$.

- Then, $X_T = o_P\left(\mathsf{a}_T\right)$ stands for

$$\lim_{T\to\infty} P\left(\left|\frac{X_T}{\mathsf{a}_T}\right| \ge \epsilon\right) = 0$$

for every $\epsilon > 0$.

- Besides, $X_T = \mathcal{O}_P(\mathsf{a}_T)$ signifies that for any $\epsilon > 0$ there exist $0 < \mathsf{N}$ and $T_0 < \infty$ such that it holds

$$P\left(\left|\frac{X_T}{\mathsf{a}_T}\right| > \mathsf{N}\right) < \epsilon$$

  for all $T > T_0$.

- Similarly, for $o_{P^\star}$ and $\mathcal{O}_{P^\star}$ we replace P by $P^\star$ in the definitions above. In addition to that, convergence and boundedness hold only in P-probability.

Lastly, the expression card$(\cdot)$ stands for cardinality. Thereby, the preparatory part is closed and we can start our journey through the worlds in the following chapter.

2 A Functional Central Limit Theorem for Locally Stationary Processes

In this chapter, we begin our journey in the first world, namely the *real world*. At the end, we aim to state a FCLT for locally stationary processes. But before we are able to do that, we have to start at the beginning. This includes the determination of our setup consisting of some definitions and assumptions. In association therewith, we shine a light on the basic idea of locally stationary processes and its origins. This takes place in the first section of this chapter. Subsequently, we will establish a central limit theorem (CLT) in the second section in preparation for the FCLT in the last one.

A particularity of this thesis is the reference to the results established in Jentsch et al. (2020), which runs like a golden thread through the chapters. In said joint work, empirical characteristic functions-based estimation for locally stationary processes was established. In the progress, a corresponding FCLT was proposed as well. Therefore, we will regard the setup in Jentsch et al. (2020) as a special case of our theory, which has its reason in the circumstance that a generalization of the results therein was the motivation for the choice of subject for this thesis. We will put a finer point on said generalization in the upcoming section.

2.1. Preliminaries

This section lays the foundation for the upcoming results. We begin with a presentation of our underlying process. Thereby, we give a short overview of local stationarity. While presenting our assumptions, we will always draw the comparison to the ones used in Jentsch et al. (2020). Afterwards, we focus more on the weights and the function belonging to the setup in Jentsch et al. (2020). This will be the starting position for our generalizations, which we present during the course of this section.

To begin with, we take a look at the history of locally stationary processes. The idea of approximating non-stationary time series on segments by stationary ones can be found by Priestley (1965), whereas the concept of local stationarity, on which this thesis is based, goes back to Dahlhaus (1997). There, the definition of local stationarity was introduced, and this conception opened the way for momentous theory via the use of an asymptotic (in-fill) framework. An overview of the state of the art is to be found in Dahlhaus (2012). Despite the name, locally stationary processes are globally non-stationary but can locally be approximated well enough by stationary ones. Since it is common in the literature to define a locally stationary process via a linear MA(∞)-representation, this will be the case in this thesis as well. Therefore, let $(\underline{\varepsilon}_t)_{t\in\mathbb{Z}}$ be a sequence of i.i.d. centered random vectors and $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ with a d-variate time series process having a time-varying MA(∞)-representation, that is

$$\underline{X}_{t,T} = \underline{\mu}\left(\frac{t}{T}\right) + \sum_{j\in\mathbb{Z}} A_{t,T}(j)\,\underline{\varepsilon}_{t-j}, \tag{2.1}$$

where $\underline{\mu}(\cdot) = (\mu_1(\cdot), \dots, \mu_d(\cdot))'$ is a time-varying mean function and $(A_{t,T}(j))_{j\in\mathbb{Z}}$ are coefficient matrices of dimension $(d\times d)$. Note that the sequence $(\underline{\varepsilon}_t)_{t\in\mathbb{Z}}$ is not necessarily a white noise because $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ is only having a MA(∞)-representation and is not automatically an MA(∞)-process as stated in Definition 1.8. To ensure that the aforedefined process exists, we need the sum in (2.1) to be finite, which means the coefficient matrices $(A_{t,T}(j))_{j\in\mathbb{Z}}$ have to decay in a sufficiently fast manner as $|j|$ tends to ∞. Additionally, we do not allow for fast variation of the coefficients over time to be able to propose useful statistical methodology.

This leads to the following assumptions concerning the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$:

Assumption 1 (Process)**.**
The process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ is of form (2.1) with the following specifications:

(i) The innovations $(\underline{\varepsilon}_t)_{t\in\mathbb{Z}}$ are i.i.d., centered and have finite first absolute moments.

(ii) There exists a real-valued, deterministic sequence $(l(j))_{j\in\mathbb{Z}}$ with

$$\sum_{j\in\mathbb{Z}} \frac{|j|^{\frac{(1+\tilde{\delta})(1+\delta)}{\delta}}}{l(j)} < \infty \tag{2.2}$$

for some $\tilde{\delta}\in(0,1)$ and a constant $B<\infty$ such that

$$\sup_{t,T} |A_{t,T}(j)|_1 \leq \frac{B}{l(j)}. \tag{2.3}$$

Further, for each $j\in\mathbb{Z}$ there exists an entry-wise continuously differentiable function $A(\cdot,j)\colon [0,1]\to\mathbb{R}^{d\times d}$ such that for all $p,q = 1,\dots,d$ and $s = 0,\dots,k$, $k\in\{0,1\}$, it holds

$$\sup_u \left|\frac{\partial^s a^{(p,q)}(u,j)}{\partial u^s}\right| \leq \frac{B}{l(j)} \tag{2.4}$$

and

$$\sup_{t,T} T \left| A_{t,T}(j) - A\left(\frac{t}{T}, j\right) \right|_1 \leq \frac{B}{l(j)}, \tag{2.5}$$

where $A(u,j)$ and $A'(u,j)$ have the building types $A(u,j) = \left(a^{(p,q)}(u,j)\right)_{p,q=1,\ldots,d}$ and $A'(u,j) = \frac{\partial}{\partial u} A(u,j) = \left(\frac{\partial}{\partial u} a^{(p,q)}(u,j)\right)_{p,q=1,\ldots,d}$, respectively.

(iii) Each component of the mean function $\underline{\mu}$ is continuously differentiable.

Remark 2.1.

(i) With Assumption 1, for the most part, we take up the assumptions made in Jentsch et al. (2020). A prominent exception is part (ii), where the exponent is now $\frac{(1+\tilde{\delta})(1+\delta)}{\delta}$ instead of 2 as in Jentsch et al. (2020). But since both δ and $\tilde{\delta}$ are valued between 0 and 1, we have the following relation:

$$\frac{\left(1+\tilde{\delta}\right)(1+\delta)}{\delta} > 2. \tag{2.6}$$

So the exponent used here is a heavier requirement, whose fulfilment implies the fulfilment of the condition imposed in Jentsch et al. (2020).

(ii) Again, similar to Jentsch et al. (2020), requisites with reference to continuity and differentiability on $[0,1]$ stated in Assumption 1 are to be understood in the one-sided sense when it comes to the boundary points.

(iii) Mainly, our assumptions correspond to those in Section 4 in Dahlhaus (2012). Instead of bounded variation of the mean process and bounded total variation of the coefficients $(A(\cdot,j))_{j\in\mathbb{Z}}$, we demand smoothness, which is more handy in statistics. In Dahlhaus (2012), smoothness assumptions are also made for estimation results and since our main focus is on estimation, we require those from the beginning. Our assumptions differ as well in terms of moment requirements because we ask for less then in Dahlhaus (2012), where finite second moments of the innovations are presupposed. Contrary to that, we stick with first absolute moments for the innovations. Later, when it comes to tightness results, we will augment our moment requirements. Nevertheless, this does not happen directly for the innovations but regarding some kind of generalized equivalent. However, as long as we abide by a setup comparable to the one Dahlhaus (2012) used, we stay under finite second moments.

(iv) As in Jentsch et al. (2020) already explained, the construction using $A_{t,T}(j)$ and $A(t/T,j)$, which are named *close pair* by Cardinali and Nason (2010), has its benefits despite its complicated looking appearance. The latter function, $A(t/T,j)$, has its good in rescaling and is needed to impose smoothness conditions, whereas the former enriches the class of processes. Examples for those are time-varying autoregressive (tvAR) processes fulfilling

$$X_{t,T} = r_{t,T}(1)\, X_{t-1,T} + \cdots + r_{t,T}(p)\, X_{t-p,T} + \varepsilon_t$$

for $t = 1, \ldots, T$ or, more general, the class of tvARMA processes. Further explanation can be found in Dahlhaus (2012).

The conditions called for in Assumption 1 represent a classical framework for statistical interference having local estimators pertaining to locally stationary processes as a base. Therefore, we can now introduce the concept of local approximation. In short, consider a fixed point in time $u = t/T$. In some neighbourhood around this point in time, the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ can be approximated locally by its so-called *companion process* $\left(\underline{\widetilde{X}}_t(u)\right)_{t\in\mathbb{Z}}$. This process is (strictly) stationary and has the following form:

$$\underline{\widetilde{X}}_t(u) = \underline{\mu}(u) + \sum_{j\in\mathbb{Z}} A(u,j)\, \underline{\varepsilon}_{t-j}. \tag{2.7}$$

Hence, compared to the original process the mean function $\underline{\mu}$ stays the same, whereas the function $A_{t,T}(j)$ is replaced by $A(u,j)$. The newly-defined $u = t/T$ is also called *rescaled time*.

Remark 2.2.
We can deduce a result concerning $\sup_{u\in[0,1]} |A(u,j)|_1$ with $j \in \mathbb{Z}$, which is comparable to equation (2.3). To this end, we avail ourselves of said equation and equation (2.5). By inserting $A\left(\frac{t}{T}, j\right) - A\left(\frac{t}{T}, j\right)$ with $t \in \{1, \ldots, T\}$, we obtain

$$|A(u,j)|_1 \le \left|A(u,j) - A\left(\frac{t}{T}, j\right)\right|_1 + \left|A\left(\frac{t}{T}, j\right) - A_{t,j}(j)\right|_1 + |A_{t,j}(j)|_1 =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

The aforementioned equations (2.3) and (2.5) imply

$$\mathrm{II} \le \frac{B}{l(j)} \quad \text{and} \quad \mathrm{III} \le \frac{B}{l(j)},$$

respectively. This leaves us with term I. Using equation (2.4), it holds

$$\begin{aligned}
\mathrm{I} &= \max_{r=1,\ldots,d} \sum_{k=1}^{d} \left| a^{k,r}(u,j) - a^{k,r}\left(\frac{t}{T}, j\right) \right| \\
&= \max_{r=1,\ldots,d} \sum_{k=1}^{d} \left| \left(\frac{\partial a^{k,r}(q,j)}{\partial q} \bigg|_{q=\xi_{k,r,j}} \right) \cdot \left(u - \frac{t}{T} \right) \right| \\
&\le \frac{dB}{l(j)}
\end{aligned}$$

for suitable $\xi_{k,r,j}$ between u and t/T. Thus, all bounds are independent of u. Hence, we conclude

$$\sup_{u\in[0,1]} |A(u,j)|_1 \leq \frac{d\widetilde{B}}{l(j)}$$

for some finite constant $\widetilde{B} > 0$. This inequality connote that (2.7) possesses a strictly stationary solution for each fixed u while Assumption 1 is satisfied.

As already mentioned above, we call for nearly the same properties to be fulfilled by our process as in Jentsch et al. (2020). Thus, we continue stating the same first consequences of Assumption 1:

Lemma 2.3 (Preliminary Consequences of Assumption 1).
Suppose that Assumption 1 holds true for $k=0$ and consider

$$C_B := \|\underline{\varepsilon}_0\|_1 \sum_{j\in\mathbb{Z}} \frac{B}{l(j)} \quad \textit{and} \quad C_{\bar{B}} := d\left(C_\mu + \|\underline{\varepsilon}_0\|_1 \sum_{j\in\mathbb{Z}} \frac{B}{l(j)}\right)$$

for some positive constant $C_\mu < \infty$ independent of t, u_1 and u_2.

(i) Then, we have

$$\sup_{1\leq t\leq T} \left\| \underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right\|_1 \leq \frac{C_B}{T}.$$

(ii) Suppose now that Assumption 1 is valid for $k=1$ as well. Then, for all $u_1, u_2 \in [0,1]$ and $t \in \mathbb{Z}$ it holds

$$\left\| \underline{\widetilde{X}}_t(u_1) - \underline{\widetilde{X}}_t(u_2)\right\|_1 \leq C_{\bar{B}} |u_1 - u_2|_1 .$$

Remark 2.4.
Other and more general definitions of local stationarity exist, see for example Vogt (2012) and Dahlhaus et al. (2018). Vogt (2012) does not require a linear representation of the process $\left(\underline{X}_{t,T}\right)_{t=1}^T$ to the price of presupposed causality. Dahlhaus et al. (2018), as well, work with causality. As this part of the thesis shall generalize the results of Jentsch et al. (2020), we stick to the requirements made there, that is a linear representation in (2.1) and high-level assumptions, to sidestep causality.

Now we have a closer look at the characteristic function (CF) and its empirical counterpart used in Jentsch et al. (2020). The characteristic function of a random vector $\underline{X}$ is defined by

$$\varphi_X(\underline{s}) := Ee^{i\langle \underline{s}, \underline{X}\rangle} \tag{2.8}$$

with $\underline{s} \in \mathbb{R}^d$. Note that the CF of a random vector always exists as the absolute value of a complex-valued exponential function is equal to 1, independently of the exponent. For given data $\underline{X}_1, \ldots, \underline{X}_T$, a classical way to estimate the CF given in (2.8) would be via the empirical characteristic function (ECF) $\widehat{\phi}_X$, that is

$$\widehat{\phi}_X(\underline{s}) = T^{-1} \sum_{t=1}^{T} e^{i\langle \underline{s}, \underline{X}_t \rangle}$$

with $\underline{s} \in \mathbb{R}^d$. The underlying idea in Jentsch et al. (2020) was to estimate $\varphi(u; \underline{s}) = Ee^{i\langle \underline{s} \widetilde{\underline{X}}_1(u) \rangle}$ based on the observations $\underline{X}_{1,T}, \ldots, \underline{X}_{T,T}$ since the CF belonging to the locally stationary process $(\underline{X}_{t,T})$ varies smoothly over time. A natural estimator for $\varphi(u; \underline{s})$ would be

$$\widehat{\varphi}_X(u; \underline{s}) = T^{-1} \sum_{t=1}^{T} e^{i\langle \underline{s}, \widetilde{\underline{X}}_t(u) \rangle},$$

but as the companion process is not observable, the CF of $\underline{X}_{\lfloor uT \rfloor, T}$ is approximated by its local sample analogue

$$\widehat{\varphi}_X(u; \underline{s}) = \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) e^{i\langle \underline{s}, \underline{X}_{t,T} \rangle}, \tag{2.9}$$

where b_T is a bandwidth depending on T and K a kernel function.

This was the framework used in Jentsch et al. (2020) in a nutshell. In Chapter 4, we will come back to this model and go more into detail. But for now, this basic description is sufficient to get an overview of the specific situation. Since this thesis wants to generalize the results shown in Jentsch et al. (2020), we can identify two aspects, where we can draw on. The first is the applied function. In Jentsch et al. (2020), the framework is restricted to the complex-valued exponential function as the focus was laid on the CF and ECF. The second aspect are the weights. In said paper, the weights consisted of a kernel function K combined with a factor $(b_T T)^{-1}$. To sum it up, in order to generalize the results we are looking at processes of the form

$$\sum_{t=1}^{T} w_{t,T}\, f(\underline{s}, \underline{X}_{t,T}) \tag{2.10}$$

with $\underline{s} \in [-S, S]^d$, a weight function $w_{t,T}$ and a function f depending on $\underline{s}$ and the locally stationary process $(\underline{X}_{t,T})_{t=1}^{T}$. This requires further assumptions on both the allowed function f and the used weights $w_{t,T}$. We start with those concerning the function:

Assumption 2 (Function).

(i) The function $f\colon [-S, S]^d \times \mathbb{R}^d \to \mathbb{R}$ has finite absolute moments of order $2 + \delta$ and

(ii) satisfies a certain Lipschitz condition specified as follows: For $\underline{s}, \underline{s}^\circ \in [-S, S]^d$ and $\mathbb{R}^d$-valued random vectors $\underline{X}, \underline{X}^\circ$, it holds

$$|f(\underline{s}, \underline{X}) - f(\underline{s}^\circ, \underline{X}^\circ)| \leq C_{Lip} |\underline{X} - \underline{X}^\circ|_1 + |\underline{s} - \underline{s}^\circ|_1\, g(\underline{X}, \underline{X}^\circ) \tag{2.11}$$

for some positive constant $C_{Lip} < \infty$ independent of the arguments of f and a function $g\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$. Moreover, the first arguments of f have no influence on g.

(iii) The absolute moments belonging to the function f can be bounded uniformly over both arguments.

Remark 2.5.
The assumptions made above seem to look rather complicated compared to the ones used in Jentsch et al. (2020). But since we do not assume the function f to be bounded, we need to impose conditions regarding its number of finite absolute moments. Besides, the real and imaginary part of the ECF are cosine and sine, respectively. Both of them fulfill the desired Lipschitz condition stated in (2.11) with $C_{Lip} = Sd$. This can easily be seen by performing the following calculations: Using a first order Taylor expansion with suitable $\underline{\xi}$ between $\underline{X}$ and $\underline{X}^\circ$ leads to

$$\begin{aligned}\left|\cos\left(\langle \underline{s}, \underline{X}\rangle\right) - \cos\left(\langle \underline{s}, \underline{X}^\circ\rangle\right)\right| &= \left|\nabla \cos\left(\langle \underline{s}, \underline{\xi}\rangle\right)\left(\underline{X} - \underline{X}^\circ\right)\right| \\ &\leq \left|\underline{s}\right|_1 \left|-\sin\left(\langle \underline{s}, \underline{\xi}\rangle\right)\right| \left|\underline{X} - \underline{X}^\circ\right|_1 \\ &\leq \left|\underline{s}\right|_1 \left|\underline{X} - \underline{X}^\circ\right|_1\end{aligned}$$

as the sine can be bounded by 1. Because $\underline{s}$ is chosen out of the range $[-S, S]^d$, it holds $\left|\underline{s}\right|_1 \leq Sd$. This yields

$$\left|\cos\left(\langle \underline{s}, \underline{X}\rangle\right) - \cos\left(\langle \underline{s}, \underline{X}^\circ\rangle\right)\right| \leq Sd \left|\underline{s}\right|_1 \left|\underline{X} - \underline{X}^\circ\right|_1 .$$

Since the sine difference can be treated analogously to the cosine one, we obtain

$$\left|\sin\left(\langle \underline{s}, \underline{X}\rangle\right) - \sin\left(\langle \underline{s}, \underline{X}^\circ\rangle\right)\right| \leq Sd \left|\underline{X} - \underline{X}^\circ\right|_1$$

for the remaining difference following a similar argumentation. On the other hand, having the same second argument we obtain

$$\begin{aligned}\left|\cos\left(\langle \underline{s}, \underline{X}\rangle\right) - \cos\left(\langle \underline{s}^\circ, \underline{X}\rangle\right)\right| &= \left|\nabla \cos\left(\langle \underline{\xi}_s, \underline{X}\rangle\right)\left(\underline{s} - \underline{s}^\circ\right)\right| \\ &\leq \left|\underline{X}\right|_1 \left|-\sin\left(\langle \underline{\xi}_s, \underline{X}\rangle\right)\right| \left|\underline{s} - \underline{s}^\circ\right|_1 \\ &\leq \left|\underline{X}\right|_1 \left|\underline{s} - \underline{s}^\circ\right|_1\end{aligned}$$

using a Taylor expansion with an appropriate $\underline{\xi}_s$ situated between $\underline{s}$ and $\underline{s}^\circ$. This means the function g plays the role of the $|\cdot|_1$-norm if its arguments are both the same. Similarly, we get

$$\left|\sin\left(\langle \underline{s}, \underline{X}\rangle\right) - \sin\left(\langle \underline{s}^\circ, \underline{X}\rangle\right)\right| \leq \left|\underline{X}\right|_1 \left|\underline{s} - \underline{s}^\circ\right|_1$$

as well. Lastly, the uniform bound required for the function f goes along with the scenario used in Jentsch et al. (2020) since both the sine and cosine function can be bounded by 1 independently of the arguments as already seen above.

Assumption 2 allows us to transfer the result seen in the first part of Lemma 2.3 to the case where the function f is applied to the local stationary process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$. To this end, we have the following lemma:

Lemma 2.6 (Preliminary Consequences of Assumption 2)**.**
Suppose the same assumption as in Lemma 2.3 is satisfied and, additionally, Assumption 2. Then, it holds

$$\sup_{\substack{1\leq t\leq T\\ \underline{s}\in[-S,S]^d}} \left\| f\left(\underline{s},\underline{X}_{t,T}\right) - f\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right\|_1 \leq \frac{C_{B'}}{T}$$

for some finite constant $C_{B'} > 0$.

After having made our requisites appurtenant to the function f, we turn our focus on the weights and demand for the following:

Assumption 3 (Weights)**.**
The weights $(w_{t,T})_{t=1}^{T}$ *fulfil the following:*

(i) The weights are non-negative and real-valued.

(ii) Let d_T^{-1} *denote the number of weights unequal to zero.*

- *It holds* $w_{t,T} \leq C_w\, d_T^{1/2}$ *for all* $t = 1, \ldots, T$ *and a finite constant* $C_w > 0$ *independent of* t.
- *The sequence* $(d_T)_{T\in\mathbb{N}}$ *satisfies* $d_T \to 0$ *as* $T \to \infty$.

Remark 2.7.
Two prominent examples are included in our choice of requirements concerning the weights. Firstly, by setting

$$w_{t,T} := (b_T T)^{-1/2}\, K\left(\frac{t/T - u}{b_T}\right)$$

for $t = 1, \ldots, T$ and $u \in [0,1]$ using a bandwidth b_T depending on T and a kernel function K we are in the ECF-case. In Chapter 4, Remark 4.1 explains more in detail that all conditions of Assumption 3 are actually satisfied. Secondly, if we deprive the weights of their dependence on t, we can obtain

$$w_{t,T} := T^{-1/2}$$

for all $t = 1, \ldots, T$, which is the prefactor used in the classical central limit theorem. The fulfillment of the requirements made in Assumption 3 is obvious in this case. Moreover, these two examples illustrate perfectly the need for the introduction of d_T^{-1} since in the first case not all of the weights are unequal to zero, whereas in the second case d_T^{-1} equals T.

The following lemma is also based on the first part of Lemma 2.3, but now, besides the function f, the weights $(w_{t,T})_{t=1}^{T}$ are included as well, that is:

Lemma 2.8.
Suppose Assumption 1 is valid for $k = 0$ and Assumptions 2 and 3 hold true. Then, for some positive constant $C_{sup} < \infty$ we have

$$E\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\left(f\left(\underline{s},\underline{X}_{t,T}\right) - f\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right)\right|\right) \leq C_{sup}\, d_T^{1/2}.$$

At this point, we will introduce some further notation to limit the complicity of the calculations following up. First, we will present a truncated version of our processes, both for the locally stationary one and the companion process, which will come in handy when independence is needed. This notation meshes with the one used in Jentsch et al. (2020). Considering $M \in \mathbb{N}$, we set

$$\underline{X}_{t,T}^{(M)} := \underline{\mu}\left(\frac{t}{T}\right) + \sum_{|j|<M} A_{t,T}(j)\,\underline{\varepsilon}_{t-j} \tag{2.12}$$

for all $t = 1, \ldots, T$ and

$$\underline{\widetilde{X}}_t^{(M)}(u) := \underline{\mu}(u) + \sum_{|j|<M} A(u,j)\,\underline{\varepsilon}_{t-j} \tag{2.13}$$

for all $u \in [0,1]$ and $t = 1, \ldots, T$. The parameter M is called *truncation parameter* hereinafter. Now we want to quantify the closeness between the locally stationary process and its truncated version. This will provide a central inequality, which will be used frequently throughout this thesis.

Lemma 2.9.
Under Assumptions 1 valid for $k = 0$ and 2, we have for every $t \in \mathbb{Z}$, $M \in \mathbb{N}$ and $u \in [0,1]$

$$\sup_{\underline{s}\in[-S,S]^d}\left\|f\left(\underline{s},\underline{\widetilde{X}}_t(u)\right) - f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}(u)\right)\right\|_1 \leq C_{Lip}\,\|\underline{\varepsilon}_0\|_1\, d\sum_{|j|\geq M}\frac{B}{l(j)}.$$

Next, we go one step further and combine the difference between the companion process and its truncated version with multiplication, since terms of this type occur in proofs where the starting point is the product of the function applied to different but not necessarily independent arguments. Then, the truncation helps to obtain independence by choosing the truncation parameter wisely. Due to the insertion of the truncated versions, terms as treated in the following lemma appear.

Lemma 2.10.
Suppose the same assumptions as in Lemma 2.9 are fulfilled. Then, it holds for all $t_1, t_2 \in \mathbb{Z}$, $M \in \mathbb{N}$ and all pairs $(\underline{s}_1, u_1), (\underline{s}_2, u_2) \in [-S,S]^d \times [0,1]$

$$\left\| \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right)\right) f\left(\underline{s}_2, \underline{\widetilde{X}}_{t_2}(u_2)\right)\right\|_1 \leq C_{DP} \left(\|\varepsilon_0\|_1 \, d \sum_{|j|\geq M} \frac{B}{l(j)} \right)^{\frac{\delta}{1-\delta}}$$

for some positive constant $C_{DP} < \infty$, which is independent of the choice of the above-mentioned pairs, t_1 and t_2 as well as M.

After having made all assumptions concerning the local stationary process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$, the applied function f and the belonging weights $(w_{t,T})_{t=1}^{T}$, we notice some differences regarding the assumptions made in Jentsch et al. (2020) as well es some additional requirements, which became necessary due to the lack of bounds concerning the function f. This leads to the use of stronger assumptions in the results we are going to establish in the following compared to the ECF-case treated in Jentsch et al. (2020). In order to show the applicability of our results under the same or even weaker requirements as in Jentsch et al. (2020) when it comes to bounded functions, we adopt a two-pronged approach. This has also the purpose to extend the results made in Jentsch et al. (2020) by leaving the path of ECFs towards bounded functions in a more general way. To this end, we propose an alternative set of assumptions when dealing with bounded functions, that is:

Assumption 4 (Assumptions Concerning a Bounded Function f)**.**

(i) Assumption 1 is satisfied, but instead of (2.2) the following condition holds:

$$\sum_{j\in\mathbb{Z}} \frac{|j|^{1+\tilde{\delta}}}{l(j)} < \infty \tag{2.14}$$

with $\tilde{\delta}$ and B as in Assumption 1.

(ii) The function $f\colon [-S,S]^d \times \mathbb{R}^d \to \mathbb{R}$, $S \in \mathbb{R}_+$ is uniformly bounded over both arguments by a positive constant $C_f < \infty$.

(iii) The Lipschitz condition (2.11) in Assumption 2 is satisfied.

Throughout the thesis, we will propose two versions of the results in most of the cases. The first version deals with an unbounded function f, whereas the second assumes a bounded f instead. However, the Lemmata 2.3, 2.6, 2.8 and 2.9 loose neither their validity nor their importance in the bounded case. This stays in contrast to Lemma 2.10. The reason why lays in the fact that said lemma is an auxiliary result tailor-made to ease the upcoming proofs regarding the unbounded function f. In the bounded case, many of the steps can be shortened by the usage of the bound to a certain extend, especially while dealing with products of f.

2.2. A Central Limit Theorem

This section's aim is the establishment of a CLT for processes of the form (2.10) as introduced in the previous section. In order to reach this goal, we will proceed in two steps. An important step of stating a CLT is the determination of the appropriate variance. Therefore, the first part of this section occupies itself with the examination of the limiting behavior of the covariance belonging to processes having the aforementioned form. Afterwards, the corresponding result for the variance can easily be deduced since it forms a special case of the then already established covariance result. This will find its use in the second part, which contains the CLT itself.

We approach the covariance belonging to the process described in (2.10) in two steps. First, we investigate the covariance of the function f applied to the companion process $\left(\underline{\widetilde{X}}_t(u)\right)_{t\in\mathbb{Z}}$. After having established a suitable bound for this covariance, we are able to handle the covariance belonging to foresaid processes. In doing so, we benefit from the fact that the locally stationary process can be replaced locally by the companion one. This explains why we direct our attention to the covariance dealing with the companion process first. Being able to use certain properties the stationary brings along, the limiting value of the covariance having the process in (2.10) as arguments can be expressed in a closed form.

Beginning with the first stage, the following lemma aims to bound the covariance dealing with the companion process but only with regard to the lag $h \neq 0$. The result cannot be broadened the way $h = 0$ is included since the lag takes part in the denominator of the bound. Nevertheless, due to the fact that f is equipped with finite $(2+\delta)$-th absolute moments, a finite bound for the variance can easily be established as well.

Lemma 2.11 (Covariance Bound).
Suppose

(a) Assumption 1 holds true for $k = 0$ and Assumption 2 is valid

or

(b) Assumption 4 is satisfied for $k = 0$.

Then, for all pairs $(\underline{s}_1, u_1), (\underline{s}_2, u_2) \in [0,1] \times [-S,S]^d$ and every $h \in \mathbb{Z}\setminus\{\underline{0}\}$ we have

$$\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right)\right| \leq \frac{C_{Cov}}{|h|^{1+\tilde{\delta}}}$$

with $\tilde{\delta}$ introduced in Assumption 1 for some positive constant $C_{Cov} < \infty$ depending neither on the above-defined pairs nor on h.

Remark 2.12.

(i) Due to stationarity, the difference between the indices of the companion process is decisive. Thus, every pair can be brought into the required form used in Lemma 2.11.

(ii) Lemma 2.11 implies the summability of the absolute values of the covariances. It holds

$$\begin{aligned}&\sum_{h\in\mathbb{Z}}\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0(u_1)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h(u_2)\right)\right)\right|\\&\leq 2\sum_{h\in\mathbb{N}}\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0(u_1)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h(u_2)\right)\right)\right|\\&\qquad\qquad+\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0(u_1)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_0(u_2)\right)\right)\right|.\end{aligned}$$

The last summand, $\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0(u_1)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_0(u_2)\right)\right)\right|$, can be bounded because of the finite absolute moments of order $2+\delta$ the function f is equipped with. In consequence, we get

$$\sum_{h\in\mathbb{Z}}\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0(u_1)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h(u_2)\right)\right)\right|\leq 2\sum_{h\in\mathbb{N}}\frac{C_{Cov}}{h^{1+\tilde{\delta}}}+C<\infty$$

since the sum over h is a geometric series.

Because the main goal of this section is the formulation of a CLT, the covariances play an important role, as the previous lemma already hinted. That means, in many cases the function f will be centered. In order to ease the notation, we will abbreviate the centered version of the function f with $\bar{f}$, that is

$$\bar{f}(\underline{s},\cdot):=f(\underline{s},\cdot)-Ef(\underline{s},\cdot) \tag{2.15}$$

for some $\underline{s}\in\mathbb{R}^d$. Throughout this thesis, further abridging notation will be introduced when needed. But for now, we confine ourselves with the notation we have on hand and turn our attention to the covariance to be used in the CLT later on.

Lemma 2.13 (Covariance I).
Suppose Assumption 1 is satisfied for $k=1$ and Assumptions 2 and 3 hold. Then, for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$ we have

$$\mathrm{Cov}\left(\sum_{t=1}^{T}w_{t,T}\,f\left(\underline{s}_1,\underline{X}_{t,T}\right),\sum_{t=1}^{T}w_{t,T}\,f\left(\underline{s}_2,\underline{X}_{t,T}\right)\right)\longrightarrow\sum_{h\in\mathbb{Z}}V_h(\underline{s}_1,\underline{s}_2)=:\sigma^2(\underline{s}_1,\underline{s}_2)$$

as $T \to \infty$ with

$$V_h\left(\underline{s}_1, \underline{s}_2\right) := \lim_{T \to \infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \tag{2.16}$$

and the limit exists.

Remark 2.14.
Lemma 2.13 stays true if we assume the validity of Assumption 4 with the given k instead of Assumptions 1 and 2. However, the proof shortens itself due to the fact that because of the boundedness of f, there is no need for the use of Hoelder's inequality. As this will only change the exponent contained in the $\mathcal{O}$-terms from $\frac{\delta}{1+\delta}$ to 1, the structure of the proof stays basically the same.

After having made the groundwork in the precedent part with the investigation of the covariance's limiting behavior, we are now able to formulate the following CLT:

Theorem 2.15 (Central Limit Theorem I)**.**
Suppose Assumption 1 holds true for $k = 1$ and Assumptions 2 and 3 are valid. Then, for $J \in \mathbb{N}$ and $\underline{s}_j \in [-S, S]^d$, $j = 1, \ldots, J$, we have

$$\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}\left(\underline{s}_j, \underline{X}_{t,T}\right), j = 1, \ldots, J\right) \xrightarrow{d} \mathcal{N}\left(\underline{0}, \mathbf{V}\right) \tag{2.17}$$

as $T \to \infty$, where $\mathbf{V} := \left(\mathbf{V}\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right)\right)_{j_1, j_2 = 1, \ldots, J}$ is a $(J \times J)$ covariance matrix with

$$\mathbf{V}\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) = \sigma^2\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right)$$

as introduced in Lemma 2.13, and $\bar{f}$ denotes the centered function as defined in (2.15).

Remark 2.16.
Theorem 2.15 holds as well if we take Assumption 4 as a base in lieu of Assumptions 1 and 2, again with the same k. As in the proof of Lemma 2.13, the proof of Theorem 2.15 is eased owing to the absence of uses of Hoelder's inequality.

2.3. A Functional Central Limit Theorem

In this section, we want to investigate the convergence in distribution of processes of the form

$$\left(\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)\right)_{\underline{s}\in[-S,S]^d}. \tag{2.18}$$

More precisely, we are interested in a distributional convergence to a Gaussian process with continuous sample path with respect to the uniform norm. The main result of this section illustrates a FCLT using the CLT established in the previous section. Such a result can be proven by combining the convergence of the fidis with a tightness result. This provides a guideline for this section. As already insinuated, the convergence of the fidis follows right away from Theorem 2.15. Thus, the establishment of said tightness result will be the first step before the formulation of the FCLT takes place.

Due to the stationarity of the companion process $\left(\widetilde{\underline{X}}_t(u)\right)_{t\in\mathbb{Z}}$, showing asymptotical tightness using this process in place of the locally stationary one $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ is easier. However, before changing to the companion process, we have to assure that the result stays valid for the original one. This is guaranteed by the following lemma:

Lemma 2.17.
Under

(a) Assumptions 1 for $k=0$, 2 and 3

or

(b) Assumption 4 for $k=0$,

it holds

$$\lim_{T\to\infty} P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)\right| > \lambda\right) = \lim_{T\to\infty} P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\widetilde{\underline{X}}_t\left(\frac{t}{T}\right)\right)\right| > \frac{\lambda}{2}\right)$$

with $\bar{f}$ denoting the centered version of the function f as in (2.15).

Now we can proceed stating the following tightness result based on the stationary companion process:

Lemma 2.18 (Tightness I)**.**
Suppose

(a) *Assumption 1 for* $k = 1$ *and Assumption 2 as well as Assumption 3 are fulfilled. However, consider*

$$\sum_{j \in \mathbb{Z}} \frac{|j|^m}{l(j)} < \infty$$

for some

$$m > \frac{20 + 15\delta - 4\delta^2 - 3\delta^3}{2\delta\left(1 - \delta^2\right)}$$

in lieu of the summation constraint (2.2). Moreover, the function g *has finite absolute* $(2+\delta)$*-th moments.*

or

(b) *Assumption 4 for* $k = 1$ *and Assumption 3 are satisfied. The exponent in the summation constraint (2.14) is replaced by some*

$$m > 1 + \frac{2}{\delta(1+\delta)}.$$

Besides, the function g *has finite absolute moments of order* $\frac{2+\delta}{2}$.

In both cases, the finite absolute moments of the function g *having the same first and second argument belonging to either the companion process or the truncated companion process can be bounded uniformly. Then, it holds for the centered function* $\bar{f}$ *as defined in (2.15)*

$$\lim_{r \to 0} \limsup_{T \to \infty} P\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} \left| \sum_{t=1}^{T} w_{t,T} \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right) \right| > \lambda \right) = 0 \tag{2.19}$$

with $\lambda > 0$ *and* $\rho(\cdot,\cdot)$ *denoting the* $|\cdot|_1$*-metric.*

At this point, solely the main result of this section is left as we are already equipped with the needed results to perform the proof written in Subsection A.1.3 of Appendix A. Thus, we proceed stating the following theorem:

Theorem 2.19 (Functional Central Limit Theorem I).
Under the validity of the same assumptions as in Lemma 2.18, version-wise, it holds

$$\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}\left(\underline{s}, \underline{X}_{t,T}\right)\right)_{\underline{s}\in[-S,S]^d} \xrightarrow{d} \left(G\left(\underline{s}\right)\right)_{\underline{s}\in[-S,S]^d}$$

as T tends to ∞, where $\left(G\left(\underline{s}\right)\right)_{\underline{s}\in[-S,S]^d}$ is a centered Gaussian process with continuous sample paths and covariance function $\sigma^2\left(\underline{s}, \underline{s}^{\circ}\right)$ originating in Lemma 2.13, whereas $\bar{f}$ denotes the centered version of f as in (2.15). In this case, convergence holds with respect to the uniform norm.

This result closes the basic real world part. In the succeeding chapter, we shift our focus to the bootstrap world.

3 A Bootstrap Functional Central Limit Theorem for Locally Stationary Processes

In this chapter, we leave the real and enter the bootstrap world. Hence, the first question to answer will be what this world is about. Afterwards, there will be a short characterization of the category of block bootstrap. Subsequently, we will present our procedure of choice and the reasons therefor. Then we go into detail with regard to the incidental algorithm and address some aspects worth noting. After the framework is depicted, we transfer our results from the real world established in Chapter 2 to the bootstrap world.

3.1. Basics and Local Block Bootstrap

Bootstrapping is a resampling method which goes back to Efron (1979). In general, resampling procedures allow for the estimation of certain parameters based on a sample set. When it comes to bootstrapping, a subset is drawn with replacement out of sample data. Afterwards, using the drawn subset the value of the sample statistic is computed. This procedure is repeated multiple times in order to obtain the empirical distribution of the sample statistic. The actual distribution, in turn, can be approximated by the empirical one. This procedure illustrates also the use of the word *world.* In the following, we will perform the calculations in the so-called *bootstrap world* based on the probability space $(\Omega^\star, \mathcal{P}(\Omega^\star), P^\star)$. Here, $\mathcal{P}(\Omega^\star)$ denotes the power set of $\Omega^\star$. Nevertheless, since the sample data originating from the real world forms the bootstrap variables, these two worlds are not fully disconnected.

This leads to the question in which way bootstrapping has to be performed to meet the purpose of its application best. Because different initial problems impose different requirements, several bootstrapping methods were developed over time. They distinguish themselves, for example, in terms of independence or correlation within the underlying sample set. Moreover, there exists a Bayesian approach as well. We will pay particular attention to the soi-disant *block bootstrap.* This type suits our starting situation quite well since we deal with dependent data. However, we have to be attentive of the local stationary characteristic. Before we go into detail, we comment on the main idea of block bootstrap. When it comes to dependent data, straightforward drawing with replacement would destroy the dependence structure. Thus, drawing blocks instead helps to conserve the correlation. The modality in which the blocks are defined denotes different types of block bootstrap. In its purest form, the simple block bootstrap, the data set is divided in blocks without any overlaps. Another method is to part the data set into blocks of the same length, but overlapping by one and rolling. After that, blocks are drawn with replacement from this set of blocks. This procedure describes sketchily the moving block bootstrap introduced by Künsch (1989) and further developed by Liu and Singh (1992). However, none of these would fit our framework since we have a locally stationary process to deal with. As described in the beginning, in this case the stochastic structure varies over time. If we pick a certain observation, the observations shortly before and after follow nearly the same distribution. An adequate block choice in order to form the block of bootstrap variables for a certain index set has to take this fact into account. Therefore, bootstrapping methods for stationary processes as considered, for example, by Lahiri (2003) cannot be applied to our framework. To cope with this problem, Paparoditis and Politis (2002) as well as Dowla et al. (2003) proposed the local block bootstrap. This methodology is based on the block bootstrap, but the peculiarity at this is the fact that the blocks are selected in such a way that their indices are close to those of the corresponding blocks of the original process. In other words, a block of a certain length L_T of the bootstrap process is formed by a block of the same length belonging to the locally stationary process whose indices are shifted by a random number. In Section 3.2, we will go more into detail while defining the bootstrap algorithm.

But for the moment, we have to return to our framework introduced in Assumption 1. To establish results in the bootstrap world, we have to modify our assumptions as we need more restrictive moment conditions in the unbounded case. Especially when it comes to covariance results, finite $(2+\delta)$-th absolute moments are not always enough. Therefore, we require the following:

Assumption 5.
Suppose Assumptions 1 for $k = 1$ plus Assumptions 2 and 3 are satisfied. In addition,

(i) instead of (2.2) it holds

$$\sum_{j\in\mathbb{Z}} \frac{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}} B}{l(j)} < \infty \tag{3.1}$$

for $\tilde{\delta} \in (0,1)$ and some positive constant $B < \infty$ as before, and

(ii) the function f has finite $(4+\delta)$-th absolute moments.

In contrast, in the case dealing with bounded f, there is no need for modification of the assumptions. On the one hand, we have as much moments as we need due to the boundedness of the function. On the other hand, since equation (3.1) is also necessary because of the changed moment requirements, this modification has no bounded counterpart either. In comparison with Assumption 4, we see that these aspects are those which differ in the real world scenario in both cases as well.

3.2. A Bootstrap Algorithm for Locally Stationary Processes

In this section, we formalize the already insinuated procedure of the local block bootstrap. The algorithm we use is based on the one proposed by Dowla et al. (2013), where it is used for trend estimation for locally stationary processes. Similar to Jentsch et al. (2020), they worked with a kernel function. Thus, the modification we make in combination with the later on established results can be seen as a generalization comparable to our proceeding in Chapter 2 regarding the situation in Jentsch et al. (2020). Our version of the local block bootstrap algorithm, which will be used as a basis throughout every bootstrap-related part of this thesis, reads as follows:

Algorithm 3.1 (Bootstrap Algorithm)**.**

(i) Consider a blocklength L_T depending on T.

- *a) Select a window parameter $D_T \in (0,1)$ such that $TD_T \in \mathbb{N}$.*
- *b) Generate i.i.d. integers $k_0, \ldots, k_{\lfloor T/L_T \rfloor -1}$ using a discrete uniform distribution on $[-TD_T, TD_T]$.*
- *c) Define $\underline{X}^\star_{1,T}, \ldots, \underline{X}^\star_{T,T}$ by*

$$\underline{X}^\star_{j+iL_T,T} := \underline{X}_{j+iL_T+k_i,T} \text{ for } j = 1, \ldots, L_T, i = 0, \ldots \lfloor T/L_T \rfloor - 1.$$

- *d) Construct the bootstrap estimator by replacing $\underline{X}_{t,T}$ with $\underline{X}^\star_{t,T}$, that is*

$$\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^\star_{t,T}\right).$$

(ii) If one selected index of a block i is outside $[1,T]$, use $-k_i$ instead of k_i for the whole block.

Remark 3.1.

(i) The window parameter D_T controls the bootstrap window, that is the range of observations being a possibility to become the bootstrap version for a fixed index.

(ii) The distribution used to generate $k_0, \ldots, k_{\lfloor T/L_T \rfloor -1}$ does not need to assign uniform weights to every choice of k_i. Here, we choose the discrete uniform distribution as it is easy to handle, analogously to Dowla et al. (2013). Besides, it matches the choice made for the original block bootstrap algorithm designed for stationary processes.

(iii) The second part of Algorithm 3.1 ensures that if a block is in danger of going over the edge, there is a sound way out. By adjusting the sign of k_i for the whole block, the interrelated dependence structure is preserved, and moreover, no observation is used twice in the same block.

The indices, which can be the reason for a change in the sign of k_i, can be grouped as in the following definition:

Definition 3.2 (Endpoints).
Considering Algorithm 3.1, all indices $t \in 1, \ldots, T$ which might cause a sign switch are called *endpoints* and can be clustered in two groups, that are

$$EP_1 := \{t \in \{1, \ldots, T\} \,|\, 1 \leq t \leq TD_T\}$$

and

$$EP_2 := \{t \in \{1, \ldots, T\} \,|\, T - TD_T < t \leq T\}\,.$$

In addition to that, the set of all endpoints, that is $\{EP_1 \cup EP_2\}$ is denoted by EP.

With the blocklength L_T and the window parameter D_T, two new parameters are involved, which need to behave good-naturedly in combination with the number of non-zero weights d_T^{-1}. Additionally, D_T is paired with T in most cases following its purpose in Algorithm 3.1. Thus, we impose the following assumptions concerning limiting and bounding behavior:

Assumption 6 (Bootstrap Rates).
For the blocklength L_T and the window parameter D_T, it holds

- $L_T \to \infty$ *as T tends to ∞ plus* $L_T = o\left(d_T^{-\frac{\delta}{2(1+\delta)}}\right)$ *and*

- $d_T^{-\frac{\delta}{2+\delta}} \leq TD_T \leq d_T^{-\frac{1}{2+\delta}}$, *that is* $\left(Td_T^{\frac{\delta}{2+\delta}}\right)^{-1} \leq D_T \leq \left(Td_T^{\frac{1}{2+\delta}}\right)^{-1}$,

respectively.

Remark 3.3.
For the sake of simplicity, we treat L_T and TD_T as positive integers for all T. This is possible without hurting Assumption 6 as there is always a sequence of numbers complying with it.

In the following sections, we aim to establish useful connections between the bootstrap expressions and those belonging to the real world. This helps us to transfer our previous findings to the bootstrap world afterwards. As a result, we will finally be able to state both a bootstrap CLT and a bootstrap FCLT. In contrast to Chapter 2, we will not divide the single results into two parts dealing either with the unbounded or the bounded version of f, respectively. This is due to the fact that the proofs differ a lot more than before. Therefore, the next two sections occupy only the unbounded case, whereas the subsequent one, namely Sections 3.5 and 3.6, concern themselves with the remaining bounded one.

But before we go into theory, we introduce some further notation. In the style of (2.15), we define

$$\bar{f}^{\star}\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) := f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) - E^{\star} f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \tag{3.2}$$

for $t = 1, \ldots, T$ and $\underline{s} \in \mathbb{R}^d$ as the bootstrap version of the centered function f. In addition to that, define

$$\bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_t(u)\right) := f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}(u)\right) - Ef\left(\underline{s}, \underline{\widetilde{X}}_t(u)\right) \tag{3.3}$$

for $t \in \mathbb{Z}$, $\underline{s} \in \mathbb{R}^d$, $u \in [0,1]$ and some $M \in \mathbb{N}$. Furthermore, $f^2(\cdot,\cdot)$ and $\bar{f}^2(\cdot,\cdot)$ are abbreviations for $(f(\cdot,\cdot))^2$ and $\left(\bar{f}(\cdot,\cdot)\right)^2$, respectively.

3.3. A Bootstrap Central Limit Theorem

This section's aim is the formulation of a bootstrap CLT. However, before we are able to do so, we have to do some preparatory work with regard to the covariance. In contrast to the real world counterpart, the preliminary results are more comprehensive. This is caused by the fact that the calculations in the bootstrap world become more complex. As a consequence thereof, the need for auxiliary results is augmented. Thus, the circuitous route to the establishment of a bootstrap CLT begins with some general preliminary results, which will become handy in several proofs in the aftermath. Nevertheless, for reasons of clarity and comprehensibility, we take most of the preparatory lemmata out of this section. Instead, they can be found in Subsection A.2.1 of Appendix A. However, the central auxiliary result, namely an upper bound for the covariance of products, is displayed in this section. Subsequently, we turn our attention to the bootstrap covariance. This leads to the P-convergence of the bootstrap variance as succeeding subject. Finally, we reach the last part of this section, which contains the bootstrap CLT.

As already mentioned, we are now also interested in the covariance of products. This leads to our first lemma:

Lemma 3.4 (Product Covariance Bound I).
Suppose Assumption 5 is satisfied.

(i) Then, for all $u \in [0,1]$, $\underline{s} \in [-S,S]^d$, $t_1 \in \mathbb{N}$ and $t_2, r \in \mathbb{N}_0$ fulfilling $t_1 > t_2$ we have

$$\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \leq \frac{C_{Cov,2i}}{(t_1 - t_2)^{1+\tilde{\delta}}}.$$

(ii) Additionally, for all $u \in [0,1]$, $\underline{s} \in [-S,S]^d$ and $t_1, t_2 \in \mathbb{N}$ with $t_1 < t_2$ it holds

$$\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+t_2}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\ \leq \begin{cases} \frac{C_{Cov,2ii}}{(t_2-t_1)^{1+\tilde{\delta}}}, & t_1 \leq \frac{t_2}{2}, \\ \frac{C_{Cov,2ii}}{t_1^{1+\tilde{\delta}}}, & t_1 > \frac{t_2}{2}. \end{cases}$$

In both cases, this holds for $\tilde{\delta}$ as given in Assumption 5 and some positive constants $C_{Cov,2i}, C_{Cov,2ii} < \infty$ independent of $\underline{s}$ and u as well as of t_1, t_2 and r, whereas $\bar{f}$ denotes the centered version of the function f as in (2.15).

Remark 3.5.
Lemma 3.4 connotes the summability of this type of covariances if the summation index is given by the difference between t_1 and t_2. For example, for $v := t_1 - t_2 > 0$ with both t_2 and r fixed, we obtain

$$\sum_{v\in\mathbb{N}}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+t_2+r}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+t_2}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\ \leq \sum_{v\in\mathbb{N}} \frac{C_{Cov,2i}}{v^{1+\tilde{\delta}}} \\ < \infty$$

as this sum portrays a geometric series.

In contradistinction to the previous result, we face the bootstrap world now. The preparatory work of Subsection A.2.1 belonging to Appendix A joint with Lemma 3.4 allows us to connect the bootstrap covariance with its real world counterpart. Since we only need this connection for index pairs which do not belong to any endpoint group, we limit ourselves to the consideration of this case.

Lemma 3.6.
Suppose Assumptions 5 and 6 are valid. Then, for all indices $t_1, t_2 \in \{1, \ldots, T\}\backslash EP$ placed in the same bootstrap block as well as for all $\underline{s} \in [-S,S]^d$ and $u \in [0,1]$ it holds

$$\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left(\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right)\right)\right. \\ \left.\cdot\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right)\right) \\ = \mathrm{Cov}\left(f\left(\underline{s}, \widetilde{X}_{t_1}(u)\right), f\left(\underline{s}, \widetilde{X}_{t_2}(u)\right)\right) + \mathcal{O}_P\left((TD_T)^{-1}\right). \tag{3.4}$$

At that, the occurring $\mathcal{O}_P$-term is independent of $t_1, t_2, \underline{s}$ and u.

Now we move on the examination of the P-boundedness of both bootstrap expectation and covariance in form of the following result:

Lemma 3.7.
Assume the validity of Assumption 5. Then, it holds for any $\underline{s} \in [-S, S]^d$

(i) for any $\underline{s} \in [-S, S]^d$ *and* $t \in \{1, \ldots, T\}$

$$E^\star f\left(\underline{s}, \underline{X}_{t,T}^\star\right) = \mathcal{O}_P(1)$$

(ii) or for any $\underline{s} \in [-S, S]^d$ *and* $t_1, t_2 \in \{1, \ldots, T\}$ *situated in the same bootstrap block*

$$\operatorname{Cov}^\star\left(f\left(\underline{s}, \underline{X}_{t_1,T}^\star\right), f\left(\underline{s}, \underline{X}_{t_2,T}^\star\right)\right) = \mathcal{O}_P(1).$$

In both cases, the $\mathcal{O}_P$*-term is not dependent on the indices or the argument* $\underline{s}$.

Progressing on our way to the statement of the bootstrap CLT, we arrive at the investigation of the P-convergence of the bootstrap variance appurtenant to the bootstrap version of (2.10). Under consideration of the previously established results, we obtain the following theorem:

Theorem 3.8 (P-Convergence of the Bootstrap Variance I)**.**
Suppose Assumptions 5 and 6 are true. Then, we have for all $\underline{s} \in [-S, S]^d$

$$\operatorname{Var}^\star\left(\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^\star\right)\right) \xrightarrow{P} \sigma^2\left(\underline{s}, \underline{s}\right)$$

as $T \to \infty$*, where* $\sigma^2\left(\underline{s}, \underline{s}\right)$ *has its roots in Lemma 2.13.*

Finally, we have passed all preparative stages to face the main result of this chapter, the bootstrap CLT, which is stated below:

Theorem 3.9 (Bootstrap Central Limit Theorem I)**.**
Under Assumptions 5 and 6, it holds for every $\underline{s} \in [-S, S]^d$

$$\sup_{v \in \mathbb{R}} \left| P^\star\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}_{t,T}^\star\right) \leq v\right) - \Phi\left(\frac{v}{\sigma\left(\underline{s}, \underline{s}\right)}\right)\right| \xrightarrow{P} 0$$

as T *tends to* ∞*. Here,* $\bar{f}^\star$ *is defined as in equation (3.2) and* $\sigma\left(\underline{s}, \underline{s}\right) > 0$ *as in Lemma 2.13.*

This result terminates Section 3.3 and clears the way to the functional version of this CLT, which is part of the following section.

3.4. A Bootstrap Functional Central Limit Theorem

Similar to the previous section, we go on a journey through auxiliary results to finish with the statement of a bootstrap FCLT. As in the real world counterpart, we use the previously established bootstrap CLT in combination with a tightness result to prove the desired theorem. Comparable to Section 2.3, the calculations become more verbose. Anew, this inquires more preliminary results and, in this case, even stronger assumptions concerning the function g, whose roots lay in the Lipschitz condition (2.11). Furthermore, we introduce the method of *good sets*. All in all, we start with the preparatory results, move on to the tightness and end with the FCLT.

Comparable to the assumptions in the real world's tightness lemma, we need to modify our requirements regarding the function g as insinuated yet. In the beginning of this chapter, we altered already the moment conditions f has to fulfil in contrast to the real world. Thus, it is not far to seek that the prerequisites we impose on g are stronger than in Section 2.3. The following assumption illustrates to which extend this has to be done:

Assumption 7 (New Assumptions Concerning the Function g)**.**
Let the function g originating from the Lipschitz condition (2.11) has finite $(4+\delta)$-th moments,

(i) which can be uniformly bounded over all $t \in \{1, \dots, T\}$ while g having the same first and second argument belonging to the locally stationary process defined in (2.1),

(ii) and over all $t \in \mathbb{Z}$ and $u \in [0,1]$ for g possessing identical first and second arguments appurtenant to the companion process of form (2.7).

(iii) Moreover, it holds for every $t \in \{1, \dots, T\}$

$$\left\| g\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right) - g\left(\underline{\widetilde{X}}_t\left(\frac{t}{T}\right), \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right\|_{4+\delta} \leq \frac{C_g}{T},$$

where C_g denotes a finite positive constant independent of t.

Remark 3.10.
Note that part (ii) of Assumption 7 implies for all $u_1, u_2 \in [0,1]$ and every $t \in \mathbb{Z}$

$$\left\| g\left(\underline{\widetilde{X}}_t(u_1), \underline{\widetilde{X}}_t(u_1)\right) - g\left(\underline{\widetilde{X}}_t(u_2), \underline{\widetilde{X}}_t(u_2)\right)\right\|_{4+\delta} \leq C.$$

Under the validity of Assumption 7, we like to show tightness in P-probability. In other words, we aim to define a suitable subset of Ω, on which the bootstrap tightness can be verified. This particular subset has to be equipped with a real world probability tending to 1 as T tends to ∞. In our case, this subset appears to be a combination of several sup-subsets, the soi-disant *good sets*, whose probabilities fulfill all the aforesaid convergence condition. As the number of sup-subsets is finite, we can intersect them to obtain the subset we are looking for. Due to the finiteness of intersections, the probability convergences holds for the so-accrued main good set as well. The denotation *good set* has its origin in the purpose these sets have to answer. Roughly said, good sets are subsets of Ω, on which certain terms behave in a favorable manner. More precisely, during the proof of the tightness result we will arrive at some points where, for example, the functions f and g have to be bounded in a certain way, which is generally not the case. However, it can be shown that the desirable bound can be verified on a good set. Thus, to be able to work with this limiting feature, we have to constrain our calculations to the $\omega \in \Omega$ belonging to said set. This is the reason why we will show the validity of some results only on parts of Ω beginning with the already hinted bounds for f and g. Nevertheless, since these results are tailor-made to suit the proof of the tightness, we attend to them exclusively in Subsection A.2.2.

Comparable to the proof of the bootstrap tightness result established by Wieczorek (2016), the real world variance in combination with the process increments will play an important role in our tightness proof. Whereas in Wieczorek (2016) the difference between the bootstrap variance and the real world one was observed, we focus on an upper bound for the real world variance. To put a finer point on that, we consider differences of the function f, where the second argument stays the same. Using the Lipschitz condition (2.11), we aim to establish an upper bound for a special form of covariance dealing with these differences, which is determined by the first arguments' difference. The following lemma illustrates this procedure:

Lemma 3.11.
Suppose Assumptions 5, 6 and 7 are satisfied. Then, for all $\underline{s}_1, \underline{s}_2 \in [-S,S]^d$ it holds

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T} \cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \leq C_{DC} \left|\underline{s}_1 - \underline{s}_2\right|_1^{1/2}$$

for some positive constant $C_{DC} < \infty$ not depending on $\underline{s}_1, \underline{s}_2$ or t.

This lemma closes the part dedicated to preparatory work situated in this section, which clears the way for the actual tightness result. As contrasted with Section 2.3, there is no preceding lemma to justify the use of the companion process since this process does not exist in the bootstrap world. Thus, we go directly ahead and present the following lemma:

Lemma 3.12 (Bootstrap Tightness I).
Suppose Assumptions 5, 6 and 7 are fulfilled. Then, there exist subsets $(\Omega_T)_{T\in\mathbb{N}}$ of Ω with $P(\Omega_T) \to 1$ as $T \to \infty$ for which it holds

$$\lim_{r\to 0} \limsup_{T\to\infty} P^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{T} w_{t,T} \left(\bar{f}^\star\left(\underline{s}_1, \underline{X}^\star_{t,T}\right) - \bar{f}^\star\left(\underline{s}_2, \underline{X}^\star_{t,T}\right)\right) \right| > \lambda \right) = 0 \qquad (3.5)$$

with $\lambda > 0$ and $\rho(\cdot,\cdot)$ denoting the $|\cdot|_1-$metric, whereas $\bar{f}^\star$ signifies the centered version of f in the bootstrap-sense as defined in (3.2).

Following the same argumentation as in Section 2.3, the combination of bootstrap CLT and tightness result allow for the statement of the main result of this section, the FCLT:

Theorem 3.13 (Bootstrap Functional Central Limit Theorem I)**.**
Let Assumptions 5, 6 and 7 be true. Then, for $\bar{f}^\star$ as in (3.2) it holds

$$\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}^\star_{t,T}\right)\right)_{\underline{s}\in[-S,S]^d} \xrightarrow{d} \left(G\left(\underline{s}\right)\right)_{\underline{s}\in[-S,S]^d}$$

in P-probability as T tends to ∞, where $\left(G\left(\underline{s}\right)\right)_{\underline{s}\in[-S,S]^d}$ is a centered Gaussian process with continuous sample paths and covariance function $\sigma^2\left(\underline{s},\underline{s}^\circ\right)$ as defined in Lemma 2.13. Here, convergence holds with respect to the uniform norm.

This theorem finalizes the theoretical part dealing with an unbounded function f. From now onwards, we zero in on the bounded case.

3.5. A Bootstrap Central Limit Theorem for Bounded Functions

In the real world, we considered two different scenarios: one with an unbounded function f and one where f is equipped with an uniform upper bound. Now we want to transfer this two-sided approach into the bootstrap world. To keep things easy, one could apply the results for the unbounded f smoothly to a bounded one. Nevertheless, by modifying the results of the previous two sections, if necessary, to be tailor-made for a bounded function, a lot of the additional assumptions become obsolete or can be weakened. In fact, there is no need for an alteration of the requirements compared to Assumption 4 before we attend to the tightness result in the next section.

This section has the same structure as its unbounded counterpart. We start with some auxiliary results, which can partly be found only in Subsection A.2.3 of Appendix A. Then, we move on to the bootstrap covariance, which leads us to the P-convergence of the bootstrap variance immediately afterwards. The last step will be the statement of the bootstrap CLT in the case for bounded f.

Similar to Section 3.3, we start with a covariance bound result dealing with products of $\bar{f}$.

Lemma 3.14 (Product Covariance Bound II).
Suppose Assumption 4 for $k = 1$ is satisfied.

(i) Then, for all $u \in [0,1]$, $\underline{s} \in [-S,S]^d$, $t_1 \in \mathbb{N}$ and $t_2, r \in \mathbb{N}_0$ with $t_1 > t_2$ we have

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \le \frac{C_{Cov,2i,b}}{(t_1-t_2)^{1+\tilde{\delta}}}.$$

(ii) Additionally, for all $u \in [0,1]$, $\underline{s} \in [-S,S]^d$ and $t_1, t_2 \in \mathbb{N}$ fulfilling $t_1 < t_2$ it holds

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+t_2}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \le \begin{cases} \frac{C_{Cov,2ii,b}}{(t_2-t_1)^{1+\tilde{\delta}}}, & t_1 \le \frac{t_2}{2}, \\ \frac{C_{Cov,2ii,b}}{t_1^{1+\tilde{\delta}}}, & t_1 > \frac{t_2}{2}. \end{cases}$$

In both parts (i) and (ii), $\tilde{\delta}$ is the same as in Assumption 4 for some positive constants $C_{Cov,2i,b}, C_{Cov,2ii,b} < \infty$ independent of $u, \underline{s}, t_1, t_2$ and r and with $\bar{f}$ signifying the centered version of f introduced in (2.15).

Remark 3.15.
The statements made in Remark 3.5 stay valid in the case dealing with bounded f.

Up next, we aim to state a conversion result for the covariance. The following remark explains why we can adopt the one already established in Subsection 3.3:

Remark 3.16.
Lemma 3.6 stays valid under both Assumptions 3 and 4 for $k = 1$ instead of Assumption 5. The proof can be carried out in a nearly identical manner. In lieu of Lemmata A.1 and 3.4, their bootstrap analogues, to wit Lemmata A.8 and 3.14, respectively, are to use to name a difference in the proofs. The application of Hoelder's inequality is omitted due to the boundedness of the function f. This eases the proof without affecting the rates as the exponents change in a favorable way to one.

For the reason presented above, the only result left to show before approaching the bootstrap variance theorem is the counterpart for Lemma 3.7, namely the following:

Lemma 3.17.
Assume the validity of Assumptions 4 for $k = 1$ and 3. Then, it holds

(i) for all $t \in \{1, \ldots, T\}$

$$E^{\star} f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \le C_f,$$

where C_f is the constant introduced in Assumption 2, *(ii) or for all $t_1, t_2 \in \{1, \ldots, T\}$ and being part of the same bootstrap block*

$$\mathrm{Cov}^\star \left(f\left(\underline{s}, \underline{X}^\star_{t_1,T}\right), f\left(\underline{s}, \underline{X}^\star_{t_2,T}\right)\right) \leq C_{EP}$$

for some positive constant $C_{EP} < \infty$

for all $\underline{s} \in [-S, S]^d$, where neither $\underline{s}$ nor t, t_1 and t_2 have any influence on the constants.

With the previously presented results, we are now able to state a convergence result for the bootstrap variance in the bounded case:

Theorem 3.18 (P-Convergence of the Bootstrap Variance II)**.**
Suppose Assumptions 3, 4 for $k = 1$ and Assumption 6 are true. Then, we have for every $\underline{s} \in [-S, S]^d$

$$\mathrm{Var}^\star \left(\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^\star_{t,T}\right)\right) \xrightarrow{P} \sigma^2\left(\underline{s}, \underline{s}\right)$$

as $T \to \infty$ with $\sigma^2\left(\underline{s}, \underline{s}\right)$ as defined in Lemma 2.13.

Lastly, we turn our attention to the main result of this section in form of the bootstrap CLT, which reads as follows:

Theorem 3.19 (Bootstrap Central Limit Theorem II)**.**
Under Assumptions 3 and 4 for $k = 1$ as well as Assumption 6 it holds for every $\underline{s} \in [-S, S]^d$

$$\sup_{v \in \mathbb{R}} \left| P^\star \left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}^\star_{t,T}\right) \leq v \right) - \Phi\left(\frac{v}{\sigma\left(\underline{s}, \underline{s}\right)}\right)\right| \xrightarrow{P} 0$$

as T tends to ∞ with $\bar{f}^\star$ introduced in (3.2) and $\sigma\left(\underline{s}, \underline{s}\right) > 0$ in Lemma 2.13.

This closes the first section dealing with a bounded function f. The next one aligns itself with this section and approaches the bounded bootstrap FCLT.

3.6. A Bootstrap Functional Central Limit Theorem for Bounded Functions

In this section, we will trace the path already predestined in Section 3.4. Beginning with an altered assumption concerning the function g, we move on to the statement a basic result for the determination of a suitable good set in order to be able to establish a tightness result. Finally, we will close this section with the statement of the FCLT.

Because we draw closer to the tightness result, we are in need of further assumptions regarding the function g. The clarifications of these prerequisites read as follows:

Assumption 8 (Additional Assumptions Concerning the Function g)**.**
Additionally to Assumption 4, it holds the following for the function g:

(i) The number of finite absolute moments belonging to g accounts for $\frac{2+\delta}{2}$.

(ii) It holds for every $t \in \{1, \dots, T\}$

$$\left\| g\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right) - g\left(\underline{\widetilde{X}}_t\left(\frac{t}{T}\right), \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right\|_{\frac{2+\delta}{2}} \leq \frac{C_{g,1}}{T},$$

(iii) and with $u_1, u_2 \in [0,1]$*, we have for every* $t \in \mathbb{Z}$

$$\left\| g\left(\underline{\widetilde{X}}_t(u_1), \underline{\widetilde{X}}_t(u_1)\right) - g\left(\underline{\widetilde{X}}_t(u_2), \underline{\widetilde{X}}_t(u_2)\right)\right\|_{\frac{2+\delta}{2}} \leq C_{g,2},$$

where both $C_{g,1}$ *and* $C_{g,2} > 0$ *are finite positive constants independent of* t, u_1 *as well as* u_2*, respectively.*

Remark 3.20.
The assumptions made above fit perfectly the ECF-case, which works as base in Jentsch et al. (2020) because the exponential function fulfils exactly what we required in Assumption 8 if the innovations possess finite $\left(\frac{2+\delta}{2}\right)$-th absolute moments. As mentioned yet, the identity provided with the absolute value plays the role of g having identical first and second arguments. Additionally, since our δ may take values between 0 and 1, whereas the corresponding parameter in Jentsch et al. (2020) is restricted between 0 and 1/2, the number of required finite absolute moments used to prove the tightness is the same. Lastly, we have part (ii) and (iii) of Assumption 8 fulfilled in the ECF-case as well. This can easily be seen by the following calculations comparable to those made in the proof of Lemma 2.3. It holds

$$\left\| \left|\underline{X}_{t,T}\right|_1 - \left|\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right|_1 \right\|_{\frac{2+\delta}{2}} \leq \sum_{j\in\mathbb{Z}} \frac{1}{T} \sup_{t,T} T \left| A_{t,T}(j) - A\left(\frac{t}{T}, j\right)\right|_1 \|\underline{\varepsilon}_0\|_{\frac{2+\delta}{2}} \leq \frac{C_1}{T}$$

using (2.5). Besides, we have

$$\begin{aligned}
&\left\| \left|\underline{\widetilde{X}}_t(u_1)\right|_1 - \left|\underline{\widetilde{X}}_t(u_2)\right|_1 \right\|_{\frac{2+\delta}{2}} \\
&\leq \left\| \underline{\widetilde{X}}_t(u_1) - \underline{\widetilde{X}}_t(u_2) \right\|_{\frac{2+\delta}{2}} \\
&\leq \left|\underline{\mu}(u_1) - \underline{\mu}(u_2)\right|_1 + \sum_{j\in\mathbb{Z}} \left\| \left(A(u_1, j) - A(u_2, j)\right)\underline{\varepsilon}_{t-j}\right\|_{\frac{2+\delta}{2}} \\
&\leq C_\mu\, d\, |u_1 - u_2| + \sum_{j\in\mathbb{Z}} |A(u_1,j) - A(u_2,j)|_1 \|\underline{\varepsilon}_0\|_{\frac{2+\delta}{2}} \\
&\leq d\left(C_\mu + \|\underline{\varepsilon}_0\|_{\frac{2+\delta}{2}} \sum_{j\in\mathbb{Z}} \frac{B}{l(j)}\right)|u_1 - u_2| \\
&\leq C_2.
\end{aligned}$$

Clearly, these inequalities correspond exactly to those we stated in Assumption 8.

Comparable to the unbounded version of this section, for the determination of suitable good sets we refer to Subsection A.2.4 of Appendix A. But before we advance to the tightness result, we establish the bounded counterpart of Lemma 3.11, that is:

Lemma 3.21.
Suppose Assumptions 3, 4 for $k = 1$ and 8 are satisfied. Then, for all $\underline{s}_1, \underline{s}_2 \in [-S, S]^d$ it holds

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$\leq \bar{C}_{DC} \left|\underline{s}_1 - \underline{s}_2\right|_1^{1/2}$$

for some finite constant $\bar{C}_{DC} > 0$ independent of the choice of $\underline{s}_1$ and $\underline{s}_2$.

In this subsection, we bring the beforehand identified good sets into use by showing the following tightness result:

Lemma 3.22 (Bootstrap Tightness II)**.**
Let Assumptions 3, 4 for $k = 1$, 6 and 8 be valid. Then, there exist subsets $\left(\bar{\Omega}_T\right)_{T\in\mathbb{N}}$ of Ω with $P\left(\bar{\Omega}_T\right) \to 1$ as $T \to \infty$ for which

$$\lim_{r\to 0} \limsup_{T\to\infty} P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - \bar{f}^\star\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right)\right| > \lambda\right) = 0, \qquad (3.6)$$

holds with $\lambda > 0$, $\rho(\cdot,\cdot)$ denoting the $|\cdot|_1$-metric and $\bar{f}^\star$ as introduced in (3.2).

Conclusively, we are now up to present the FCLT stated below:

Theorem 3.23 (Bootstrap Functional Central Limit Theorem II)**.**
Suppose the validity of Assumptions 3, 4 for $k = 1$, 6 and 8. Then, with $\bar{f}^\star$ denoting the centered version of f in the bootstrap-sense as seen in (3.2) it holds

$$\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}_{t,T}^\star\right)\right)_{\underline{s}\in[-S,S]^d} \xrightarrow{d} \left(G\left(\underline{s}\right)\right)_{\underline{s}\in[-S,S]^d}$$

in P-probability as $T \to \infty$, where $\left(G\left(\underline{s}\right)\right)_{\underline{s}\in[-S,S]^d}$ is a centered Gaussian process with continuous sample paths and covariance function $\sigma^2\left(\underline{s}, \underline{s}^\circ\right)$ as defined in Lemma 2.13. In this scenario, convergence holds with respect to the uniform norm.

Comparable to Chapter 2, this result marks the end of the investigation of the basic bootstrap theory. In the following chapter, this theory in combination with the results originating from Chapter 2 will find its application to statistics.

4 A Bootstrap-focused Simulation Study

In this chapter, we want to fill the results we obtained in the previous ones with life by performing various simulations. Since those simulations require the selection of a function f to apply the results to, we draw the line back to the setup used in Jentsch et al. (2020). In other words, we return to the ECF-case. Moreover, this entails the need for additional bootstrap results, which broadens our theory to a later on specified case in particular. The structure of this chapter is built as follows: We start with the theoretical part, which deals with the ECF in general. In the middle part of this chapter, we choose an underlying distribution for the locally stationary process and illustrate our established results with different simulations. Finally, the last section shows the simulation results and presents our interpreting thoughts about them.

4.1. Additional Theory

This section attends to the specialized theory we need to conduct our simulations. We begin by stating our assumptions regarding the weights more precisely. In this way, they are tailor-made for the ECF-case. Afterwards, we establish some supplementary results, whose requirement will be explained shortly afterwards.

Assumption 9 (Kernel and Bandwidth).
The kernel function and the bandwidth fulfill the following conditions:

(i) The function $K\colon \mathbb{R} \to [0, \infty)$ is non-negative, symmetric and Lipschitz continuous with Lipschitz constant $C_{Lip,K}$. Besides, it integrates up to 1 and has compact support $[-1, 1]$.

(ii) The sequence of bandwidths $(b_T)_{T\in\mathbb{N}}$ is non-negative and fulfills $b_T \to 0$ and $b_T^2 T \to \infty$ as T tends to ∞.

Remark 4.1.

(i) Since the kernel is non-negative, the weights composed by said kernel and the prefactor $(b_T T)^{1/2}$ used in the ECF-case inherit the non-negativity.

(ii) The number of weights unequal to 0 is determined by the kernel function K. Since there is compact support on $[-1, 1]$, the argument has to fulfill

$$-1 \leq \frac{t/T - u}{b_T} \leq 1,$$

which is equivalent to

$$T (u - b_T) \leq t \leq T (u + b_T),$$

to render a positive value for K. This results in at most $2 \lfloor b_T T \rfloor + 1$ non-zero weights occurring in a sum from 1 to T. Thus, the inverse of the number of non-zero weights tends to 0 as T tends to ∞.

(iii) The kernel has to be bounded due to its Lipschitz continuity and the compact support. Joint with the prefactor $(b_T T)^{1/2}$, the kernel meets the bounding requirements regarding the weights in Assumption 3 as well.

(iv) With the previous parts in mind, we have for the blocklength

$$L_T = o\left((b_T T)^{\frac{\delta}{2(1+\delta)}}\right).$$

In the upcoming proofs, we are often in a situation where the difference between a weighted sum of the kernel and 1 has to be quantified. The following lemma potters at this subject:

Lemma 4.2.
Suppose Assumption 9 holds true. Then, we have

$$\left| \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) - 1 \right| = \mathcal{O}\left(\frac{1}{b_T T}\right).$$

The question which comes in mind is why there would be any need for supplementary results if the previous chapters were a generalization of the ECF-case. While this question is a reasonable one, the answer is rather pellucid: The generalization interrelates either real or imaginary parts of $\widehat{\varphi}_X (u; \underline{s})$ and $E\, \widehat{\varphi}_X (u; \underline{s})$, whereas we are interested in a relation between the corresponding parts of both $\widehat{\varphi}_X (u; \underline{s})$ and $\varphi_X (u; \underline{s})$. Thus, we need to provide the connecting results, which are not generalizable since they are case-specific. In this section, we will break the first ground because for our simulation studies, we do not need all of the binding results. Later on, in Section 5.2, we will continue with their establishment. Returning to this section, we will close eventually by meeting the capability of the construction of confidence intervals. But first, we start with a result which quantifies the difference between $E\,\widehat{\varphi}_X\,(u;\underline{s})$ and $\varphi_X\,(u;\underline{s})$. The following result can also be found in Jentsch et al. (2020) as a conclusion of another result therein. Nevertheless, our assumptions differ as it can be seen by considering Assumption 4.

Lemma 4.3.
Suppose Assumption 4 is fulfilled for $k = 1$ and additionally Assumption 9. Moreover, it holds $b_T^3 T = o(1)$. Then, we have for all $u \in [0,1]$

$$\sup_{\underline{s}\in[-S,S]^d} (b_T T)^{1/2}\, \Re\left(E\,\widehat{\varphi}_X\,(u;\underline{s}) - \varphi_X\,(u;\underline{s})\right) = o(1)$$

and

$$\sup_{\underline{s}\in[-S,S]^d} (b_T T)^{1/2}\, \Im\left(E\,\widehat{\varphi}_X\,(u;\underline{s}) - \varphi_X\,(u;\underline{s})\right) = o(1),$$

where the o-terms are independent of the choice of u.

At this point, we are able to reformulate Theorem 3.19 at our convenience. The new version manifests itself as follows:

Theorem 4.4.
Let the same assumptions as in Lemma 4.3 be satisfied. Additionally, suppose Assumption 6 holds true. Then, it holds for every $\underline{s} \in [-S,S]^d$

$$\sup_{v\in\mathbb{R}} \Big| P^\star\left((b_T T)^{1/2}\, \Re\left(\widehat{\varphi}_X^\star\,(u;\underline{s}) - E^\star\widehat{\varphi}_X^\star\,(u;\underline{s})\right) \leq v\right) \\ - P\left((b_T T)^{1/2}\, \Re\left(\widehat{\varphi}_X\,(u;\underline{s}) - \varphi_X\,(u;\underline{s})\right) \leq v\right)\Big| \xrightarrow{P} 0$$

and

$$\sup_{v\in\mathbb{R}} \Big| P^\star\left((b_T T)^{1/2}\, \Im\left(\widehat{\varphi}_X^\star\,(u;\underline{s}) - E^\star\widehat{\varphi}_X^\star\,(u;\underline{s})\right) \leq v\right) \\ - P\left((b_T T)^{1/2}\, \Im\left(\widehat{\varphi}_X\,(u;\underline{s}) - \varphi_X\,(u;\underline{s})\right) \leq v\right)\Big| \xrightarrow{P} 0$$

as T tends to ∞.

Remark 4.5.
As a direct consequence of Theorem 4.4, we are now able to formulate bootstrap confidence intervals for both $\Re\varphi_X\,(u;\underline{s})$ and $\Im\varphi_X\,(u;\underline{s})$ by the pivotal method without relying on the normal distribution. Worth mentioning is that Theorem 3.19 would have already made bootstrap confidence intervals possible for both $\Re E\,\widehat{\varphi}_X\,(u;\underline{s})$ and $\Im E\,\widehat{\varphi}_X\,(u;\underline{s})$ in this manner.

4.2. α-stable Distributions and Setup

The central section of Chapter 4 quits the theoretical path in favor of a more practical view. In other words, we illustrate our findings by a simulation study in this section. As indicated yet, we tie in with the ECF considered in Jentsch et al. (2020). Like already explained therein, the CF is helpful in cases where moment-based methods cannot be applied. However, as opposed to the aforementioned paper, we change the emphasis and focus on the bootstrap procedure. In doing so, we investigate the coverage at a significance level of 0.05 for $(b_T T)^{1/2}\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)$ while using bootstrap quantiles in place of their real world counterparts. At it, we consider the empirical 0.025- and the empirical 0.975-bootstrap quantile to obtain the desired significance level. To generate our process, we stick to the class of α-stable distributions, which were already chosen in Jentsch et al. (2020) as foundation. Therefore, at first we will highlight the most important reasons why said distributions suit us well. Subsequently, we elaborate on the actual parameter choice. Note that during the simulation study, we choose the dimension to be $d = 1$.

A distribution belonging to the class of α-stable distributions is determined by four parameters, namely the location parameter $\mu \in \mathbb{R}$, the characteristic exponent $\alpha \in (0, 2]$, the skewness parameter $\beta \in [-1, 1]$ and the scale parameter $\gamma \geq 0$, and the corresponding CF

$$\varphi_{\mu,\alpha,\beta,\gamma}(s) = \exp\left(i\mu s - \gamma|s|^{\alpha}\left(1 + i\beta \operatorname{sgn}(s)\,\tau(s,\alpha)\right)\right) \tag{4.1}$$

for $s \in \mathbb{R}$ with

$$\tau(s,\alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right), & \alpha \neq 1, \\ \frac{2}{\pi}\log|s|, & \alpha = 1. \end{cases}$$

This class is very versatile as it contains both light and heavy tailed distributions, which can be seen by considering two special cases: First, with characteristic exponent $\alpha = 2$ we obtain a normal distribution. Moreover, by choosing $\alpha = 1$ and $\beta = 0$ we find ourselves in the subclass of Cauchy distributions. Besides, the use of $\alpha < 2$ and $\beta \neq 0$ leads to asymmetric distributions.

Returning to our situation, α-stable distributions have a helpful advantage: If the innovations $\left(\underline{\varepsilon}_t\right)_{t\in\mathbb{Z}}$ follow an α-stable distribution with parameters $(\alpha, \beta, \gamma, \mu)$, linear processes $\left(\underline{\widetilde{X}}_t(u)\right)_{t\in\mathbb{Z}}$ like in equation (2.7) remain α-stable with parameters

$$\begin{aligned}
\widetilde{\alpha}(u) &= \alpha, \\
\widetilde{\mu}(u) &= \mu, \\
\widetilde{\beta}(u) &= \beta\frac{\sum_{j\in\mathbb{Z}}\operatorname{sgn}\left(a(u,j)\right)\,|a(u,j)|^{\alpha}}{\sum_{j\in\mathbb{Z}}|a(u,j)|^{\alpha}}, \\
\widetilde{\gamma}(u) &= \gamma\sum_{j\in\mathbb{Z}}|A(u,j)|^{\alpha}.
\end{aligned}$$

Consequently, the process $\left(\underline{\widetilde{X}}_t(u)\right)_{t\in\mathbb{Z}}$ has the following time-varying CF:

$$\varphi(u;s) = \varphi_{\widetilde{\mu}(u),\widetilde{\alpha}(u),\widetilde{\beta}(u),\widetilde{\gamma}(u)}(s) = \exp\left(i\widetilde{\mu}(u)s - \widetilde{\gamma}(u)|s|^{\widetilde{\alpha}(u)}\left(1 + i\widetilde{\beta}(u)\operatorname{sgn}(s)\,\widetilde{\tau}\left(s,\widetilde{\alpha}(u)\right)\right)\right) \tag{4.2}$$

for any $s \in \mathbb{R}$ with

$$\widetilde{\tau}\left(s,\widetilde{\alpha}(u)\right) = \begin{cases} \tan\left(\frac{\pi\widetilde{\alpha}(u)}{2}\right), & \widetilde{\alpha}(u) \neq 1, \\ \frac{2}{\pi}\log|s|, & \widetilde{\alpha}(u) = 1, \end{cases}$$

according to (4.1). This enables us to model time-varying α-stable distributions. Finally, to tie in with the combination of α-stable distributions and both CFs and ECFs, we have a look at the available moments. Roughly speaking, a process following an α-stable distribution for $\alpha < 2$ disposes of every number of finite absolute moments less than α. Thus, second absolute finite moments are rare and even first ones are not always the case. This renders the use of moment-based methods difficult.

In the following, we present our simulation setup, which is the foundation for the results presented in the subsequent section. As already mentioned, we restrict ourselves to the dimension $d = 1$. The goal is to obtain tables of coverage results to examine the impact of different bootstrap parameter choices.

The locally stationary process $(X_{t,T})_{t=1}^{T}$ is generated according to

$$X_{t,T} = 0.9\sin\left(2\pi\frac{t}{T}\right)X_{t-1,T} + \varepsilon_t$$

for $t = 2,\ldots,T$. For $t = 1$, in turn, we set

$$X_{1,T} = 0.9\sin\left(2\pi\frac{1}{T}\right)\varepsilon_0 + \varepsilon_1.$$

By this means, $(X_{t,T})_{t=1}^{T}$ is a time-varying centered AR(1)-process, which, according to Dahlhaus (2012) can be approximated locally by a stationary process. Said companion process has the following form:

$$\widetilde{X}_t(u) = \sum_{j\in\mathbb{N}_0}\left(0.9\sin(2\pi u)\right)^j\varepsilon_{t-j}.$$

Seizing on the class of α-stable distributions described above, the innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ form an i.i.d. sequence and follow an α-stable marginal distribution, whose CF as seen in equation (4.1) is determined by

$$\mu = 0,\ \alpha = 1.5,\ \beta = 0 \quad \text{and} \quad \gamma = 0.5. \tag{4.3}$$

Thus, we have centered innovations and the marginal distribution is symmetric. This leads us directly to the question which values for δ are possible in this setup. As yet explained above, in case of a symmetric α-stable distribution with α being smaller than 2 we have all finite absolute moments smaller than α. Hence, the innovations possess finite absolute moments of order $\frac{2+\delta}{2}$ for every $\delta \in (0,1)$. As explained before, the companion process inherits the belonging to the distribution class but with altered parameters. More precisely, the parameter choice as in (4.3) translates to

$$\widetilde{\mu}(u) = 0,\ \widetilde{\alpha}(u) = 1.5,\ \widetilde{\beta}(u) = 0 \quad \text{and} \quad \widetilde{\gamma}(u) = \frac{0.5}{1 - |0.9\sin(2\pi u)|^{1.5}}.$$

Inserting these parameters into equation (4.2) yields

$$\varphi(u;s) = \exp\left(-\frac{0.5|s|^{1.5}}{1 - |0.9\sin(2\pi u)|^{1.5}}\right)$$

as belonging CF for the companion process. Up to this point, our setup corresponds to the one proposed in Jentsch et al. (2020). Inspired by Dowla et al. (2013), we adopt the notation using δ_1, δ_2 and δ_3 and move on with the determination of both the bandwidth and the bootstrap rates. To be more precise, we set

$$L_T = \left\lfloor T^{\delta_1} \right\rfloor,\ TD_T = \left\lfloor T^{1-\delta_2} \right\rfloor \quad \text{and} \quad b_T = \left\lfloor T^{-\delta_3} \right\rfloor. \tag{4.4}$$

Next, we have to choose δ_1, δ_2 and δ_3 in such a way Assumptions 6 and 9 are satisfied. In order to examine the influence of said choices, we identify different possible combinations for the three parameters, which are intertwined by the bandwidth. This is the reason why we take the choice for δ_3 as point of departure for the determination of both δ_1 and δ_2. Because bandwidth optimization is a whole field on its own, we do not claim our choices for δ_3 to be optimal. Instead, they show possibilities. Since L_T and D_T are the bootstrap-related parameters, and we are willing to show the performance of our bootstrap procedure, we are more interested in which way we can enhance the results by changing the values of δ_1 and δ_2. From Assumption 9 we know the bandwidth has to fulfill $b_T^3 T = o(1)$, while $b_T^2 T$ still tends to ∞ as T tends to ∞. Therefore, we choose either $\delta_3 = 0.35$ or $\delta_3 = 0.4$, which both meet the aforesaid prerequisites. Before we can go on to identify possible candidates for the remaining two parameters, we have to stipulate the value for δ. Since there are no boundaries other than 0 and 1 due to $\alpha = 1.5$, we opt for both $\delta = 0.45$ and $\delta = 0.75$ and investigate both cases individually. Comparable to the bandwidth, we do not claim these choices to be optimal or especially good in a certain way. They only represent two possibilities which are not situated very close to each other with respect to the interval $(0,1)$. Now we are able to determine values for δ_2 and δ_3. Starting with the pair consisting of $\delta = 0.45$ and $\delta_3 = 0.35$, we obtain the constraint

$$L_T < T^{0.65 \cdot \frac{0.45}{2 \cdot 1.45}}$$

according to Assumption 6 if we disregard any constants. This leads to $\delta_1 \leq 0.1$. Next, we focus on the window parameter D_T. Inserting our choices for δ and δ_3 into upper and lower bound leads to

$$T^{0.65 \cdot \frac{0.45}{2.45}} \leq T^{0.12}$$

and

$$T^{0.65 \cdot \frac{1}{2.45}} \geq T^{0.26}.$$

Consequently, we have $0.74 \leq \delta_2 \leq 0.88$. Swapping the value for δ_3 while leaving δ equal to 0.45, we obtain $\delta_1 \leq 0.09$ and $0.76 \leq \delta_2 \leq 0.88$. Since we are interested in the impact a change of the parameter choice has, it is sensible to choose the values for δ_1 and δ_2 in such a way a direct comparison is allowed. In other words, we pick out values which are possible in both versions for δ_3. In our case, we settle for $\delta_1 \in \{0.06, 0.09\}$ and $\delta_2 \in \{0.76, 0.82, 0.88\}$. Repeating the same steps for $\delta = 0.75$, we obtain again ranges of possible values. To gain even more comparability, we decide upon the same values as before, if possible. Thus, we have $\delta_1 \in \{0.06, 0.09, 0.12\}$ and $\delta_2 \in \{0.79, 0.82\}$. Note that a larger value for δ allows for larger values for δ_1, but for a smaller interval in respect of δ_2. After the determination of combinations for δ as well as δ_1, δ_2 and δ_3, there is one parameter left to be particularized, namely u. Since in Jentsch et al. (2020) different results were obtained in the simulations while using different choices of u, we will most likely observe the same effect. Nevertheless, we have to appoint one value for u to perform our simulations with. Because our results established in the previous section deal with the real and imaginary part of the ECF individually, our simulations will adopt this procedure. This is why we obtain both the coverage based on the real part and one resting upon the imaginary part. However, a coverage based on the absolute value of a CF difference will be stated as well. Notwithstanding, this third coverage does not affect our choice for the value of u used in the simulations. On the contrary, we use box plots based on either the real or imaginary part of the ECF to identify a good choice for u. The values we are interested in are

$$(b_T T)^{1/2} \Re \left(\widehat{\varphi}_X^\star(u; s) - E^\star \widehat{\varphi}_X^\star(u; s) \right) \tag{4.5}$$

and

$$(b_T T)^{1/2} \Im \left(\widehat{\varphi}_X^\star(u; s) - E^\star \widehat{\varphi}_X^\star(u; s) \right), \tag{4.6}$$

respectively.

Following the setup in Jentsch et al. (2020), we evaluate the bootstrap simulation for $u \in \{0.2, 0.4, 0.6, 0.8\}$ and $s = 0.6$. The bootstrap parameters will be $\delta_1 = 0.12, \delta_2 = 0.79$ and $\delta_3 = 0.35$, which indicates a δ equal to 0.75. The sample size will be $T = 100000$, whereas the number of bootstrap repetitions will be $N_{BS} = 50000$ to impair the influence of outliers. The sample size seems to be relatively large, but the simulation study will eventually back it up.

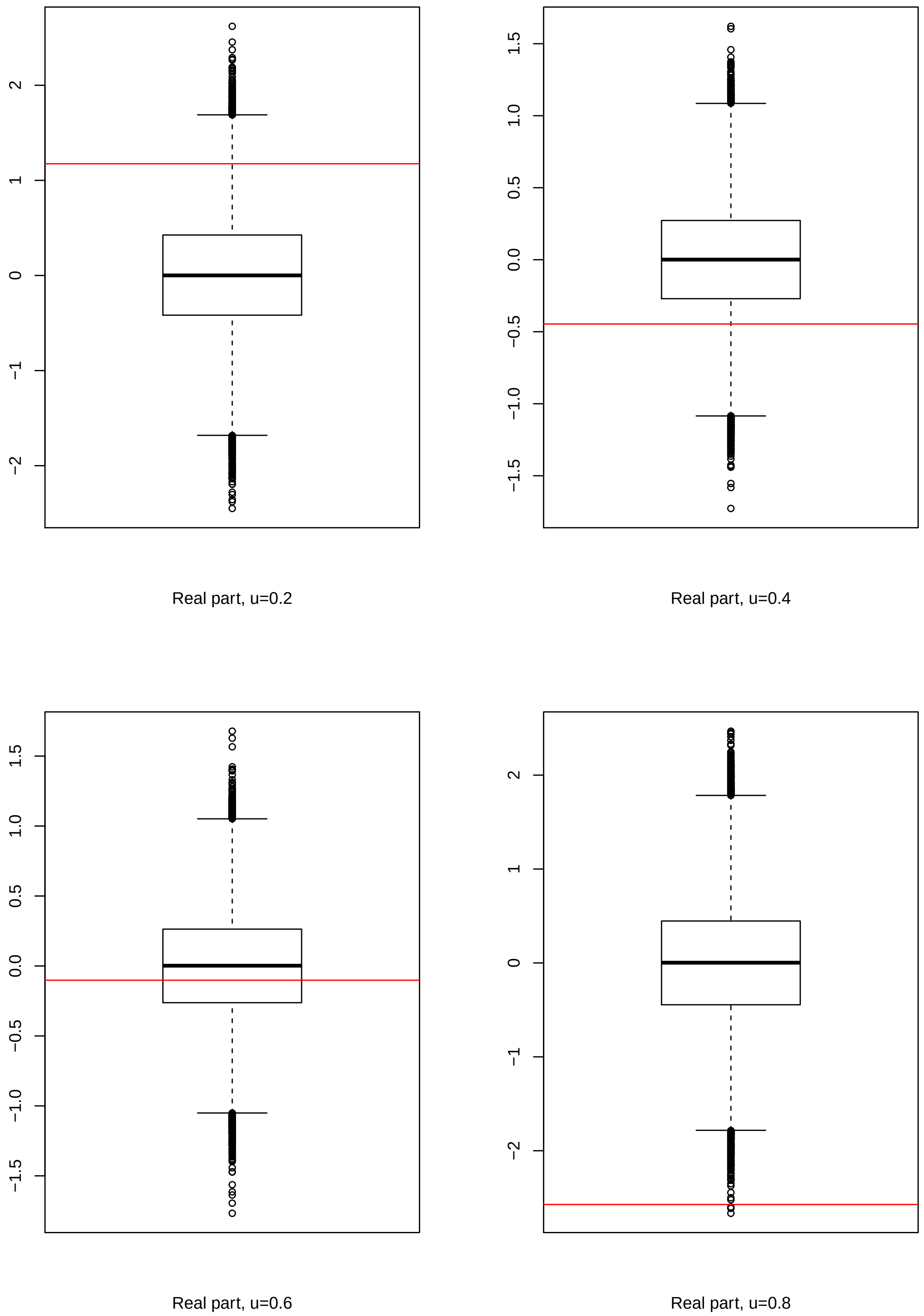

Figure 4.1: Box plots for $(b_T T)^{1/2}\,\Re\left(\widehat{\varphi}_X^{\star}(u;s) - E^{\star}\,\widehat{\varphi}_X^{\star}(u;s)\right)$ based on $s = 0.6$ $\delta_1 = 0.12, \delta_2 = 0.79$ and $\delta_3 = 0.35$ as well as $T = 100000$, whereas the number of bootstrap repetitions accounts for $N_{BS} = 50000$.

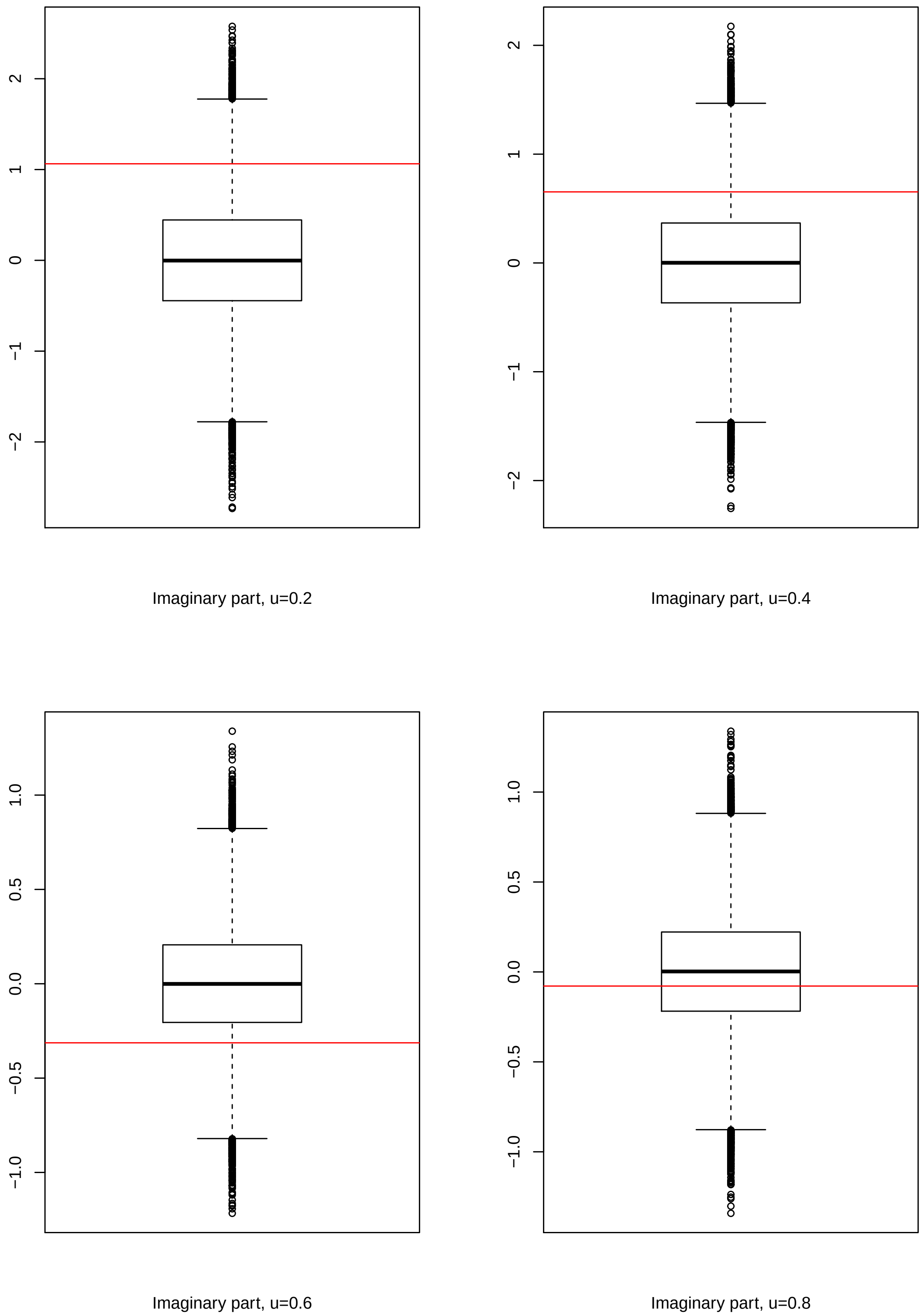

Figure 4.2: Box plots for $(b_T T)^{1/2}\,\Im\left(\widehat{\varphi}^{\star}_X(u;s) - E^{\star}\,\widehat{\varphi}^{\star}_X(u;s)\right)$ based on $s = 0.6$ $\delta_1 = 0.12, \delta_2 = 0.79$ and $\delta_3 = 0.35$ as well as $T = 100000$, whereas the number of bootstrap repetitions accounts for $N_{BS} = 50000$.

Figures 4.1 and 4.2 show the box plots of the differences stated in equations (4.5) and (4.6), respectively. The red line indicates the real or imaginary part of the weighted difference between ECF and CF. As we can clearly see, $u = 0.2$ does not seem to be a good choice regarding both the real and imaginary part. The same counts for $u = 0.8$ when it comes to the real part in Figure 4.1. Although the imaginary case as displayed in Figure 4.2 is completely different, this value is not a favorable choice for u since the red line is not even visible in Figure 4.1. Between the remaining two values, $u = 0.6$ fares slightly better. Nevertheless, we will consider both possibilities for our simulations as the results are both way better than those for $u \in \{0.2, 0.8\}$. This can especially be seen in the real part in Figure 4.1 by focussing on the statistical dispersion.

We conduct our simulation study for a consistent number of $N = 1000$ repetitions with $N_{BS} = 2000$ bootstrap repetitions within while varying the sample size. The smallest will be $T = 1000$ and the largest $T = 1000000$ with four increments between. Moreover, we stick to $s = 0.6$.

Due to our choice for the bandwidth and the block length, we do not have to consider endpoints. They are filtered out by the kernel function. As an example, we consider $T = 1000, \delta_1 = 0.09, \delta_2 = 0.82, \delta_3 = 0.35$ and $u = 0.4$. Then, the smallest index which is not sifted out by the kernel function is $t = 311$ and the largest is $t = 489$ while having a blocklength of $L_T = 1$ and a bootstrap window size of 7. Similar calculus can be performed for the other combinations of parameters as well.

Another aspect worth mentioning relates to the bootstrap quantiles. We use the empirical bootstrap quantiles generated by N_{BS} repetitions of the bootstrap procedure rather than the ones given by the bootstrap distribution.

Now we are prepared to conduct the simulation study, whose results will be presented in the following section.

4.3. The Simulation Results

This section contains the tables resulting from our simulations according to the setup described in the previous section. Additionally, we discuss the results and highlight peculiarities.

For each parameter combination, there are four different results listed, to wit three different coverages and, separated by a dashed line, an explanatory one. The three coverage results are named $\mathcal{C}_{\Re}, \mathcal{C}_{\Im}$ and $\mathcal{C}_{\text{abs}}$. The first two belong to the real and the imaginary part of $(b_T T)^{1/2} \left(\widehat{\varphi}_X(u; s) - \varphi_X(u; s)\right)$, respectively, whereas the last is appurtenant to the absolute value of said term. The remaining result, denoted by $\mathcal{C}_{\text{joint}}$, represents a combination of both $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$, which means every time the expression in question is covered by the empirical bootstrap confidence interval in both the real and the imaginary case, it is captured by a counter, which outputs $\mathcal{C}_{\text{joint}}$ in the end. Because this counter is also scaled down by N, its value cannot exceed the smaller one out of $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$.

In total, we examine ten tables. The first four belong to $u = 0.4$. At it, we differentiate between the choices for δ and δ_3, whereas the varied values for δ_1 and δ_2 are contained in each table itself. The first half of these tables refers to $\delta = 0.45$, and the second addresses $\delta = 0.75$.

$u = 0.4, \delta = 0.45$		δ_3=0.35					
		δ_2=0.76		δ_2=0.82		δ_2=0.88	
		δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09
$T = 1000$	$\mathcal{C}_{\Re}$	0.702	0.702	0.671	0.671	0.626	0.626
	$\mathcal{C}_{\Im}$	0.647	0.647	0.607	0.607	0.566	0.566
	$\mathcal{C}_{\text{abs}}$	0.611	0.611	0.574	0.574	0.505	0.505
	$\mathcal{C}_{\text{joint}}$	0.467	0.467	0.414	0.414	0.362	0.362
$T = 10000$	$\mathcal{C}_{\Re}$	0.771	0.830	0.763	0.797	0.763	0.761
	$\mathcal{C}_{\Im}$	0.718	0.761	0.673	0.746	0.644	0.656
	$\mathcal{C}_{\text{abs}}$	0.729	0.787	0.657	0.748	0.631	0.649
	$\mathcal{C}_{\text{joint}}$	0.552	0.634	0.524	0.591	0.490	0.508
$T = 20000$	$\mathcal{C}_{\Re}$	0.798	0.836	0.744	0.820	0.732	0.755
	$\mathcal{C}_{\Im}$	0.696	0.780	0.698	0.743	0.616	0.691
	$\mathcal{C}_{\text{abs}}$	0.712	0.797	0.673	0.757	0.592	0.684
	$\mathcal{C}_{\text{joint}}$	0.556	0.646	0.519	0.610	0.459	0.519
$T = 50000$	$\mathcal{C}_{\Re}$	0.807	0.857	0.788	0.826	0.742	0.770
	$\mathcal{C}_{\Im}$	0.712	0.782	0.688	0.746	0.651	0.683
	$\mathcal{C}_{\text{abs}}$	0.712	0.792	0.682	0.752	0.626	0.693
	$\mathcal{C}_{\text{joint}}$	0.579	0.673	0.545	0.616	0.486	0.531
$T = 100000$	$\mathcal{C}_{\Re}$	0.802	0.853	0.775	0.837	0.757	0.780
	$\mathcal{C}_{\Im}$	0.680	0.808	0.694	0.764	0.622	0.686
	$\mathcal{C}_{\text{abs}}$	0.702	0.826	0.700	0.769	0.619	0.671
	$\mathcal{C}_{\text{joint}}$	0.546	0.688	0.530	0.635	0.473	0.528
$T = 1000000$	$\mathcal{C}_{\Re}$	0.865	0.896	0.867	0.880	0.828	0.831
	$\mathcal{C}_{\Im}$	0.817	0.852	0.794	0.831	0.751	0.776
	$\mathcal{C}_{\text{abs}}$	0.844	0.861	0.815	0.858	0.762	0.783
	$\mathcal{C}_{\text{joint}}$	0.709	0.768	0.686	0.792	0.618	0.647

Table 4.1: Coverage results based on $u = 0.4, \delta = 0.45$ and $\delta_3 = 0.35$ together with $s = 0.6$ as well as $N = 1000$ and $N_{BS} = 2000$.

At this point, we have an initial glance at the coverage results based on $u = 0.4$. The first finding is a quite natural one. With increasing sample size, the coverage increases as well. Since our results are mostly valid for T tending to ∞, this is quite reasonable. The biggest differences are between $T = 1000$ and $T = 10000$ as well as $T = 100000$ and $T = 1000000$, which is both a decupling of the sample size. Looking at the first two coverage results belonging to one parameter constellation and one sample size, we see always a gradation. At it, the higher number appertains to the real part. The difference

$u = 0.4, \delta = 0.45$		δ_3=0.4					
		δ_2=0.76		δ_2=0.82		δ_2=0.88	
		δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09
$T = 1000$	$\mathcal{C}_{\Re}$	0.744	0.744	0.692	0.692	0.674	0.674
	$\mathcal{C}_{\Im}$	0.656	0.656	0.618	0.618	0.597	0.597
	$\mathcal{C}_{\text{abs}}$	0.653	0.653	0.563	0.563	0.557	0.557
	$\mathcal{C}_{\text{joint}}$	0.490	0.490	0.428	0.428	0.403	0.403
$T = 10000$	$\mathcal{C}_{\Re}$	0.793	0.830	0.786	0.803	0.718	0.760
	$\mathcal{C}_{\Im}$	0.688	0.770	0.686	0.758	0.643	0.697
	$\mathcal{C}_{\text{abs}}$	0.696	0.790	0.670	0.764	0.622	0.705
	$\mathcal{C}_{\text{joint}}$	0.545	0.645	0.534	0.612	0.474	0.532
$T = 20000$	$\mathcal{C}_{\Re}$	0.781	0.859	0.774	0.813	0.731	0.757
	$\mathcal{C}_{\Im}$	0.714	0.803	0.685	0.760	0.651	0.693
	$\mathcal{C}_{\text{abs}}$	0.711	0.831	0.677	0.761	0.643	0.697
	$\mathcal{C}_{\text{joint}}$	0.559	0.689	0.532	0.627	0.479	0.530
$T = 50000$	$\mathcal{C}_{\Re}$	0.801	0.863	0.788	0.828	0.749	0.763
	$\mathcal{C}_{\Im}$	0.721	0.824	0.677	0.761	0.648	0.701
	$\mathcal{C}_{\text{abs}}$	0.743	0.832	0.696	0.788	0.617	0.688
	$\mathcal{C}_{\text{joint}}$	0.573	0.708	0.541	0.636	0.492	0.539
$T = 100000$	$\mathcal{C}_{\Re}$	0.787	0.835	0.782	0.834	0.736	0.761
	$\mathcal{C}_{\Im}$	0.709	0.794	0.700	0.785	0.646	0.675
	$\mathcal{C}_{\text{abs}}$	0.703	0.827	0.700	0.793	0.614	0.675
	$\mathcal{C}_{\text{joint}}$	0.550	0.666	0.544	0.659	0.483	0.523
$T = 1000000$	$\mathcal{C}_{\Re}$	0.874	0.894	0.869	0.879	0.812	0.836
	$\mathcal{C}_{\Im}$	0.824	0.852	0.802	0.829	0.747	0.769
	$\mathcal{C}_{\text{abs}}$	0.839	0.878	0.823	0.858	0.760	0.791
	$\mathcal{C}_{\text{joint}}$	0.715	0.764	0.702	0.736	0.610	0.636

Table 4.2: Coverage results based on $u = 0.4, \delta = 0.45$ and $\delta_3 = 0.4$ together with $s = 0.6$ as well as $N = 1000$ and $N_{BS} = 2000$.

between real and imaginary part is attributable to the use of either the cosine or the sine function with their specific properties, for example symmetry. In most cases, the value for $\mathcal{C}_{\text{abs}}$ stands between the aforementioned ones. Thus, we can assume some countervailing effects. A significant exception are the results for $T = 1000$. In all but one case, the coverage rate for the absolute value of the CF-difference is smaller than both the ones for real and imaginary part. Since this phenomenon disappears for larger sample sizes, the comparably small value for T might be the reason for it. Out of all four results, the smallest belongs to the combination of $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$. However, it is trivial that the two input coverages both surpass $\mathcal{C}_{\text{joint}}$ as they are both smaller than 1. Now we focus on the relation between $\mathcal{C}_{\text{abs}}$ and $\mathcal{C}_{\text{joint}}$. In the single case for $T = 1000$ where the value for $\mathcal{C}_{\text{abs}}$ is not smaller than the two other coverage rates, the value for $\mathcal{C}_{\text{joint}}$ is comparatively large. On the other hand, for sample sizes greater than 1000 with $\mathcal{C}_{\text{abs}}$ being smaller than both $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$ we observe a relatively small $\mathcal{C}_{\text{joint}}$. This indicates a certain coherence between $\mathcal{C}_{\text{abs}}$ and $\mathcal{C}_{\text{joint}}$. Moreover, there seems to be the following connection: the comparatively smaller $\mathcal{C}_{\text{joint}}$ is the smaller is $\mathcal{C}_{\text{abs}}$ as well. Consequently, we observe some compensatory effects, but the benefit from higher convergence rates of real and imaginary part outweighs them.

Now we focus on Tables 4.1 and 4.2 and turn our attention to the impacts the different choices for δ_2 and δ_3 have. At this, we observe better results with increasing bootstrap window sizes induced by δ_2 and larger blocklengths determined by δ_1. Of course, this cannot be extended at random because of the constraints given to make the procedure work. Focusing on δ_1, there are two peculiarities. First, looking at the sample size $T = 1000$, both choices for δ_1 induce the same blocklength, namely $L_T = 1$. This leads us directly to the second point. A blocklength equal to 1 overrides the procedure of block bootstrap and the advantages it brings along. Hence, the results are worse. Lastly, the differences between $\delta_3 = 0.35$ and $\delta_3 = 0.4$ are marginal without a clearly dominant parameter choice. Overall, a coverage of nearly 0.9 is reached for the real part, whereas the one appurtenant to the imaginary part does not reach 0.86. This results in a combined coverage of slightly less than 0.8 at most. For the coverage of the absolute value, we obtain a slightly larger value that 0.86 at most. Compared with the underlying significance level of $0.05 = 1 - 0.95$, the results for the real part are quite well, while the other two coverage results are improvable. Now we move on to the tables dealing with $\delta = 0.75$.

In addition to what we have seen in Tables 4.1 and 4.2, we notice there is no difference between the results using either $\delta_1 = 0.06$ or $\delta_1 = 0.09$ considering $T = 1000$. The reason behind is the fact that both choices for δ_1 induce the same blocklength, namely $L_T = 1$. If we augment the value for δ_1 to be 0.12, we cross the border to obtain a blocklength of 2, which results in better coverage outcomes. Moreover, the results for larger δ_1 strengthens our impression provoked by the first two tables. Returning to the differences between $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$, we notice that, given $T = 1000$ and $T = 1000000$, the difference is smaller than in the other sample size cases. This effect was not as visible in Tables 4.1 and 4.2. For the real part, the coverage results exceed 0.9, whereas the highest rate for $\mathcal{C}_{\Im}$ increases by roughly 0.01. Additionally, $\mathcal{C}_{\text{abs}}$ reaches 0.9 as well. However, the combined value is not improved.

Before we move on to $u = 0.6$, we have a detailed look at the absolute values for the kernel window size, the blocklength and the bootstrap window size to obtain a better understanding of the results. For this purpose, consider $u = 0.4$ and $\delta = 0.75$. Then, Table 4.5 shows how much the effective sample size is downsized by the kernel function. The larger T becomes the more serious are the differences. Nevertheless, this has not as much impact on the coverage results as one would expect after seeing the numbers. However, it is remarkable that different value combinations for T and δ_3 produce comparable kernel window sizes, for example $T = 10000$ combined with $\delta_3 = 0.35$ in comparison to $T = 20000$ joint with $\delta_3 = 0.4$. To investigate the effects of this phenomenon on the coverage results, we consider $\delta_1 = 0.9$ and $\delta_3 = 0.82$ while u is still fixed on 0.4. Table 4.6 consolidates kernel window size and coverage results belonging to the real part. Looking closely at said table, we notice that for every pair the coverage rate listed on the second place is the higher one, whereas the kernel window is simultaneously larger only in the second case. In the remaining ones, it is exactly the opposite. In conclusion, not the effective sample size after the application of the kernel function is decisive for the coverage but

$u=0.4, \delta=0.75$		δ_3=0.35					
		δ_2=0.79			δ_2=0.82		
		δ_1=0.06	δ_1=0.09	δ_1=0.12	δ_1=0.06	δ_1=0.09	δ_1=0.12
$T=1000$	$\mathcal{C}_{\Re}$	0.684	0.684	0.726	0.671	0.671	0.705
	$\mathcal{C}_{\Im}$	0.633	0.633	0.693	0.607	0.607	0.652
	$\mathcal{C}_{\text{abs}}$	0.591	0.591	0.671	0.574	0.574	0.606
	$\mathcal{C}_{\text{joint}}$	0.439	0.439	0.503	0.414	0.414	0.463
$T=10000$	$\mathcal{C}_{\Re}$	0.765	0.840	0.844	0.763	0.797	0.825
	$\mathcal{C}_{\Im}$	0.672	0.714	0.783	0.673	0.746	0.769
	$\mathcal{C}_{\text{abs}}$	0.673	0.743	0.818	0.657	0.748	0.794
	$\mathcal{C}_{\text{joint}}$	0.519	0.596	0.656	0.524	0.591	0.626
$T=20000$	$\mathcal{C}_{\Re}$	0.760	0.846	0.853	0.744	0.820	0.825
	$\mathcal{C}_{\Im}$	0.685	0.771	0.819	0.698	0.743	0.782
	$\mathcal{C}_{\text{abs}}$	0.676	0.782	0.838	0.673	0.757	0.797
	$\mathcal{C}_{\text{joint}}$	0.525	0.650	0.702	0.519	0.610	0.642
$T=50000$	$\mathcal{C}_{\Re}$	0.765	0.836	0.860	0.788	0.826	0.869
	$\mathcal{C}_{\Im}$	0.693	0.770	0.801	0.688	0.745	0.822
	$\mathcal{C}_{\text{abs}}$	0.676	0.783	0.823	0.682	0.752	0.833
	$\mathcal{C}_{\text{joint}}$	0.561	0.648	0.685	0.545	0.616	0.712
$T=100000$	$\mathcal{C}_{\Re}$	0.795	0.873	0.852	0.775	0.837	0.851
	$\mathcal{C}_{\Im}$	0.724	0.781	0.835	0.694	0.764	0.802
	$\mathcal{C}_{\text{abs}}$	0.721	0.808	0.845	0.700	0.769	0.833
	$\mathcal{C}_{\text{joint}}$	0.570	0.681	0.716	0.530	0.635	0.684
$T=1000000$	$\mathcal{C}_{\Re}$	0.847	0.873	0.904	0.867	0.880	0.861
	$\mathcal{C}_{\Im}$	0.819	0.844	0.867	0.794	0.831	0.840
	$\mathcal{C}_{\text{abs}}$	0.815	0.872	0.888	0.815	0.858	0.869
	$\mathcal{C}_{\text{joint}}$	0.691	0.735	0.782	0.686	0.729	0.727

Table 4.3: Coverage results based on $u=0.4, \delta=0.75$ and $\delta_3=0.35$ together with $s=0.6$ as well as $N=1000$ and $N_{BS}=2000$.

the actual one. Besides, the bootstrap window size does not grow with the same potential as the sample size or even the shrunken sample size. Regarding the blocklength, this phenomenon is even stronger, for example, concerning $\delta_1=0.06$, the difference between $T=1000$ and $T=1000000$ is 1. Moreover, we see that there is no difference between the blocklength induced by $\delta_1=0.06$ or $\delta_1=0.09$ in the case $T=1000$. This reflects the results we got in Tables 4.3 and 4.4.

Now we proceed with the coverage tables and consider $u=0.6$ in lieu of 0.4. This gives the results listed in Tables 4.7 to 4.10. We start our examination with the first two, that are those using $\delta=0.45$.

The first thing which strikes the eye is the fact that $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$ turned the tables. Now the value for the latter exceeds the one belonging to the former. Moreover, the value for $\mathcal{C}_{\Im}$ is constantly larger than 0.95. Thus, our confidence interval seems to be too conservative in the imaginary case. As a result, the combined coverage is higher as in the former cases.

$u=0.4, \delta=0.75$		δ_3=0.4					
		δ_2=0.79			δ_2=0.82		
		δ_1=0.06	δ_1=0.09	δ_1=0.12	δ_1=0.06	δ_1=0.09	δ_1=0.12
$T=1000$	$\mathcal{C}_{\Re}$	0.731	0.731	0.762	0.692	0.692	0.734
	$\mathcal{C}_{\Im}$	0.654	0.654	0.712	0.618	0.618	0.663
	$\mathcal{C}_{\text{abs}}$	0.638	0.638	0.715	0.563	0.563	0.662
	$\mathcal{C}_{\text{joint}}$	0.478	0.478	0.540	0.428	0.428	0.495
$T=10000$	$\mathcal{C}_{\Re}$	0.781	0.810	0.842	0.786	0.803	0.826
	$\mathcal{C}_{\Im}$	0.680	0.781	0.794	0.686	0.758	0.773
	$\mathcal{C}_{\text{abs}}$	0.676	0.790	0.820	0.670	0.764	0.794
	$\mathcal{C}_{\text{joint}}$	0.526	0.627	0.673	0.534	0.612	0.643
$T=20000$	$\mathcal{C}_{\Re}$	0.778	0.865	0.863	0.774	0.813	0.805
	$\mathcal{C}_{\Im}$	0.708	0.775	0.798	0.685	0.760	0.746
	$\mathcal{C}_{\text{abs}}$	0.721	0.802	0.819	0.677	0.761	0.769
	$\mathcal{C}_{\text{joint}}$	0.568	0.671	0.690	0.532	0.627	0.602
$T=50000$	$\mathcal{C}_{\Re}$	0.779	0.830	0.858	0.788	0.828	0.846
	$\mathcal{C}_{\Im}$	0.704	0.767	0.819	0.677	0.761	0.770
	$\mathcal{C}_{\text{abs}}$	0.706	0.795	0.843	0.696	0.788	0.792
	$\mathcal{C}_{\text{joint}}$	0.544	0.633	0.701	0.541	0.636	0.658
$T=100000$	$\mathcal{C}_{\Re}$	0.793	0.879	0.869	0.782	0.834	0.842
	$\mathcal{C}_{\Im}$	0.717	0.801	0.835	0.700	0.785	0.794
	$\mathcal{C}_{\text{abs}}$	0.721	0.808	0.845	0.700	0.769	0.833
	$\mathcal{C}_{\text{joint}}$	0.578	0.704	0.727	0.544	0.659	0.666
$T=1000000$	$\mathcal{C}_{\Re}$	0.873	0.882	0.900	0.869	0.879	0.885
	$\mathcal{C}_{\Im}$	0.833	0.842	0.858	0.802	0.829	0.839
	$\mathcal{C}_{\text{abs}}$	0.845	0.865	0.890	0.823	0.858	0.867
	$\mathcal{C}_{\text{joint}}$	0.782	0.743	0.772	0.702	0.736	0.747

Table 4.4: Coverage results based on $u=0.4, \delta=0.75$ and $\delta_3=0.4$ together with $s=0.6$ as well as $N=1000$ and $N_{BS}=2000$.

$u=0.4$, $\delta=0.75$	kernel window		bootstrap window	blocklength		
	δ_3=0.35	δ_3=0.4	δ_2=0.82	δ_1=0.06	δ_1=0.09	δ_1=0.12
$T=1000$	179	127	7	1	1	2
$T=10000$	797	503	11	1	2	3
$T=20000$	1249	761	11	1	2	3
$T=50000$	2267	1319	15	1	2	3
$T=100000$	3557	2001	15	1	2	3
$T=1000000$	15887	7963	25	2	3	5

Table 4.5: Comparison of kernel and bootstrap window size plus blocklength based on $u=0.4$ and $\delta=0.75$ combined with $s=0.6$ plus $N=1000$ and $N_{BS}=2000$.

Because $\mathcal{C}_{\Im}$ is close to 1 most of the time, $\mathcal{C}_{\text{joint}}$ is almost equal to $\mathcal{C}_{\Re}$. As a consequence, its explanatory value for $\mathcal{C}_{\text{abs}}$ is lessened. As before, $\mathcal{C}_{\text{abs}}$ is situated between $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$ but always below 0.95. The ramifications caused by variations of $\delta_1, \delta_2, \delta_3$ and T are the

$u = 0.4, \delta_1 = 0.9, \delta_2 = 0.82$		kernel window	$\mathcal{C}_{\Re}$
$T = 10000,$	$\delta_3 = 0.35$	797	0.797
$T = 20000,$	$\delta_3 = 0.4$	761	0.813
$T = 20000,$	$\delta_3 = 0.35$	1249	0.820
$T = 50000,$	$\delta_3 = 0.4$	1319	0.828
$T = 50000,$	$\delta_3 = 0.35$	2267	0.826
$T = 100000,$	$\delta_3 = 0.4$	2001	0.834

Table 4.6: Kernel window size in dependence of sample and bootstrap window size based on $u = 0.4, \delta_1 = 0.9$ and $\delta_2 = 0.82$ and its impact on $\mathcal{C}_{\Re}$ with underlying $s = 0.6$ as well as $N = 1000$ and $N_{BS} = 2000$.

$u = 0.6, \delta = 0.45$		δ_3=0.35					
		δ_2=0.76		δ_2=0.82		δ_2=0.88	
		δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09
$T = 1000$	$\mathcal{C}_{\Re}$	0.743	0.743	0.698	0.698	0.658	0.658
	$\mathcal{C}_{\Im}$	0.999	0.999	0.995	0.995	0.998	0.998
	$\mathcal{C}_{\text{abs}}$	0.902	0.902	0.912	0.912	0.893	0.893
	$\mathcal{C}_{\text{joint}}$	0.742	0.742	0.693	0.693	0.657	0.657
$T = 10000$	$\mathcal{C}_{\Re}$	0.781	0.822	0.754	0.797	0.736	0.761
	$\mathcal{C}_{\Im}$	0.999	0.983	0.999	0.972	0.998	0.975
	$\mathcal{C}_{\text{abs}}$	0.911	0.885	0.926	0.854	0.927	0.828
	$\mathcal{C}_{\text{joint}}$	0.780	0.808	0.753	0.776	0.734	0.741
$T = 20000$	$\mathcal{C}_{\Re}$	0.787	0.839	0.758	0.821	0.731	0.779
	$\mathcal{C}_{\Im}$	0.998	0.972	0.997	0.975	0.998	0.967
	$\mathcal{C}_{\text{abs}}$	0.933	0.903	0.910	0.861	0.920	0.844
	$\mathcal{C}_{\text{joint}}$	0.785	0.815	0.756	0.801	0.731	0.756
$T = 50000$	$\mathcal{C}_{\Re}$	0.809	0.864	0.779	0.822	0.762	0.759
	$\mathcal{C}_{\Im}$	0.999	0.978	1.000	0.975	0.999	0.961
	$\mathcal{C}_{\text{abs}}$	0.920	0.908	0.912	0.897	0.929	0.830
	$\mathcal{C}_{\text{joint}}$	0.808	0.844	0.779	0.801	0.762	0.728
$T = 100000$	$\mathcal{C}_{\Re}$	0.815	0.856	0.768	0.848	0.732	0.733
	$\mathcal{C}_{\Im}$	0.998	0.973	0.999	0.969	0.998	0.970
	$\mathcal{C}_{\text{abs}}$	0.934	0.886	0.916	0.890	0.920	0.843
	$\mathcal{C}_{\text{joint}}$	0.814	0.831	0.768	0.823	0.730	0.749
$T = 1000000$	$\mathcal{C}_{\Re}$	0.860	0.884	0.860	0.877	0.813	0.831
	$\mathcal{C}_{\Im}$	0.966	0.988	0.981	0.972	0.977	0.971
	$\mathcal{C}_{\text{abs}}$	0.917	0.919	0.907	0.904	0.872	0.876
	$\mathcal{C}_{\text{joint}}$	0.829	0.874	0.844	0.851	0.795	0.806

Table 4.7: Coverage results based on $u = 0.6, \delta = 0.45$ and $\delta_3 = 0.35$ together with $s = 0.6$ as well as $N = 1000$ and $N_{BS} = 2000$.

same as before, although the increase of the coverage regarding the imaginary part is now a decrease towards 0.95. Nevertheless, even with decreasing values for $\mathcal{C}_{\Im}$, $\mathcal{C}_{\text{joint}}$ is still increasing due to a comparatively higher increase of $\mathcal{C}_{\Re}$. As opposed to this, such a dis-

$u=0.6,\delta=0.45$		δ_3=0.4					
		δ_2=0.76		δ_2=0.82		δ_2=0.88	
		δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09	δ_1=0.06	δ_1=0.09
$T=1000$	$\mathcal{C}_{\Re}$	0.737	0.737	0.703	0.703	0.684	0.684
	$\mathcal{C}_{\Im}$	0.997	0.997	0.999	0.999	0.997	0.997
	$\mathcal{C}_{\text{abs}}$	0.918	0.918	0.915	0.915	0.907	0.907
	$\mathcal{C}_{\text{joint}}$	0.735	0.735	0.702	0.702	0.684	0.684
$T=10000$	$\mathcal{C}_{\Re}$	0.771	0.849	0.767	0.808	0.718	0.741
	$\mathcal{C}_{\Im}$	0.999	0.975	1.000	0.973	1.000	0.969
	$\mathcal{C}_{\text{abs}}$	0.932	0.893	0.929	0.867	0.936	0.835
	$\mathcal{C}_{\text{joint}}$	0.770	0.828	0.767	0.789	0.718	0.716
$T=20000$	$\mathcal{C}_{\Re}$	0.796	0.847	0.759	0.799	0.754	0.740
	$\mathcal{C}_{\Im}$	1.000	0.968	0.998	0.972	0.997	0.964
	$\mathcal{C}_{\text{abs}}$	0.943	0.885	0.925	0.861	0.925	0.814
	$\mathcal{C}_{\text{joint}}$	0.796	0.820	0.757	0.777	0.752	0.717
$T=50000$	$\mathcal{C}_{\Re}$	0.812	0.859	0.742	0.846	0.752	0.767
	$\mathcal{C}_{\Im}$	0.998	0.974	0.998	0.981	0.999	0.973
	$\mathcal{C}_{\text{abs}}$	0.929	0.905	0.925	0.894	0.920	0.835
	$\mathcal{C}_{\text{joint}}$	0.811	0.836	0.740	0.828	0.752	0.750
$T=100000$	$\mathcal{C}_{\Re}$	0.817	0.867	0.756	0.825	0.724	0.765
	$\mathcal{C}_{\Im}$	1.000	0.972	0.999	0.969	0.997	0.963
	$\mathcal{C}_{\text{abs}}$	0.923	0.896	0.914	0.879	0.918	0.847
	$\mathcal{C}_{\text{joint}}$	0.817	0.844	0.755	0.799	0.722	0.740
$T=1000000$	$\mathcal{C}_{\Re}$	0.859	0.880	0.861	0.870	0.820	0.834
	$\mathcal{C}_{\Im}$	0.977	0.980	0.974	0.978	0.977	0.967
	$\mathcal{C}_{\text{abs}}$	0.895	0.917	0.913	0.915	0.883	0.897
	$\mathcal{C}_{\text{joint}}$	0.839	0.861	0.839	0.853	0.802	0.805

Table 4.8: Coverage results based on $u = 0.6, \delta = 0.45$ and $\delta_3 = 0.4$ together with $s = 0.6$ as well as $N = 1000$ and $N_{BS} = 2000$.

tinct trend is not observable for $\mathcal{C}_{\text{abs}}$. Overall, the maximal value for $\mathcal{C}_{\Re}$ is slightly above 0.88, whereas the best result for $\mathcal{C}_{\Im}$ is 0.966. In combination, the nearest value to 0.95 is a little less than 0.88. Lastly, for the coverage belonging to the absolute value of the CF-difference, we obtain 0.943 at the maximum, which, however, is not for $T = 1000000$.

Next, we turn to the cases addressing $\delta = 0.75$. Once again, we see the same effects as in the cases dealing with $\delta = 0.45$. Most noticeably, the roles of $\mathcal{C}_{\Re}$ and $\mathcal{C}_{\Im}$ stay changed. The highest value for $\mathcal{C}_{\Re}$ surpasses 0.9 now, whereas the best value for the coverage appurtenant to the imaginary part is 0.953. Together, both coverages reach nearly 0.88. For $\mathcal{C}_{\text{abs}}$, the maximal value falls down to 0.94.

Reviewing all tables, we notice that, taking all particularities into account, the best values are achieved for the combination of $\delta_1 = 0.12$ and $\delta_2 = 0.79$, which falls in line with our assumption of better results for both larger blocklength and larger bootstrap windows.

$u=0.6, \delta=0.75$		δ_3=0.35					
		δ_2=0.79			δ_2=0.82		
		δ_1=0.06	δ_1=0.09	δ_1=0.12	δ_1=0.06	δ_1=0.09	δ_1=0.12
$T=1000$	$\mathcal{C}_{\Re}$	0.694	0.694	0.739	0.698	0.698	0.714
	$\mathcal{C}_{\Im}$	0.999	0.999	0.960	0.995	0.995	0.965
	$\mathcal{C}_{\text{abs}}$	0.892	0.892	0.806	0.912	0.912	0.813
	$\mathcal{C}_{\text{joint}}$	0.693	0.693	0.707	0.693	0.693	0.687
$T=10000$	$\mathcal{C}_{\Re}$	0.761	0.802	0.848	0.754	0.797	0.833
	$\mathcal{C}_{\Im}$	0.998	0.962	0.968	0.999	0.972	0.964
	$\mathcal{C}_{\text{abs}}$	0.913	0.865	0.875	0.926	0.854	0.868
	$\mathcal{C}_{\text{joint}}$	0.759	0.773	0.819	0.753	0.776	0.800
$T=20000$	$\mathcal{C}_{\Re}$	0.758	0.821	0.844	0.747	0.813	0.832
	$\mathcal{C}_{\Im}$	0.997	0.975	0.977	0.999	0.968	0.966
	$\mathcal{C}_{\text{abs}}$	0.909	0.879	0.887	0.910	0.861	0.891
	$\mathcal{C}_{\text{joint}}$	0.756	0.801	0.824	0.747	0.787	0.802
$T=50000$	$\mathcal{C}_{\Re}$	0.782	0.862	0.862	0.779	0.822	0.838
	$\mathcal{C}_{\Im}$	1.000	0.963	0.972	1.000	0.975	0.972
	$\mathcal{C}_{\text{abs}}$	0.926	0.906	0.909	0.912	0.897	0.900
	$\mathcal{C}_{\text{joint}}$	0.782	0.831	0.837	0.779	0.801	0.814
$T=100000$	$\mathcal{C}_{\Re}$	0.788	0.843	0.879	0.768	0.848	0.853
	$\mathcal{C}_{\Im}$	0.999	0.976	0.971	0.999	0.969	0.977
	$\mathcal{C}_{\text{abs}}$	0.923	0.891	0.912	0.916	0.890	0.896
	$\mathcal{C}_{\text{joint}}$	0.787	0.824	0.855	0.768	0.823	0.834
$T=1000000$	$\mathcal{C}_{\Re}$	0.881	0.874	0.879	0.860	0.877	0.872
	$\mathcal{C}_{\Im}$	0.980	0.976	0.953	0.981	0.972	0.959
	$\mathcal{C}_{\text{abs}}$	0.912	0.905	0.899	0.907	0.904	0.885
	$\mathcal{C}_{\text{joint}}$	0.862	0.854	0.836	0.844	0.851	0.839

Table 4.9: Coverage results based on $u = 0.6, \delta = 0.75$ and $\delta_3 = 0.35$ together with $s = 0.6$ as well as $N = 1000$ and $N_{BS} = 2000$.

A natural extension of this study would be by taking constants into account. First, it is possible to define the bandwidth via $b_T = C_{bw}T^{-\delta_3}$ for some positive constant $C_{bw} < \infty$. However, first attempts in this direction did not show any improvement of the results but the opposite. Using constants in the definition of the bootstrap window size is more complicated as upper and lower bound are sharp instead of o-terms. Nevertheless, it is possible while taking into account that different sample sizes allow for different constants. The most promising use of a constant seems to be with regard to the blocklength, at least to prevent a blocklength of 1, which undermines the idea of block bootstrap. But this would be the foundation of another study, from which we back away due to the extend of this thesis.

Lastly, we would like to draw a comparison between our results and those achieved by Dowla et al. (2013). With our constraints in mind, which are more restrictive, our simulations perform quite well in relation to those in Dowla et al. (2013). Clearly, our sample sizes surpass the ones in said paper. However, Dowla et al. (2013) have, for example,

$u = 0.6, \delta = 0.75$		δ_3=0.4					
		δ_2=0.79			δ_2=0.82		
		δ_1=0.06	δ_1=0.09	δ_1=0.12	δ_1=0.06	δ_1=0.09	δ_1=0.12
$T = 1000$	$\mathcal{C}_{\Re}$	0.733	0.733	0.762	0.703	0.703	0.748
	$\mathcal{C}_{\Im}$	0.998	0.998	0.966	0.999	0.999	0.966
	$\mathcal{C}_{\text{abs}}$	0.908	0.908	0.863	0.915	0.915	0.838
	$\mathcal{C}_{\text{joint}}$	0.731	0.731	0.736	0.702	0.702	0.722
$T = 10000$	$\mathcal{C}_{\Re}$	0.775	0.828	0.841	0.767	0.808	0.827
	$\mathcal{C}_{\Im}$	0.998	0.976	0.970	1.000	0.973	0.965
	$\mathcal{C}_{\text{abs}}$	0.915	0.881	0.896	0.929	0.867	0.888
	$\mathcal{C}_{\text{joint}}$	0.773	0.810	0.813	0.767	0.789	0.796
$T = 20000$	$\mathcal{C}_{\Re}$	0.777	0.819	0.860	0.759	0.799	0.798
	$\mathcal{C}_{\Im}$	0.996	0.977	0.978	0.998	0.972	0.954
	$\mathcal{C}_{\text{abs}}$	0.926	0.878	0.910	0.925	0.861	0.847
	$\mathcal{C}_{\text{joint}}$	0.774	0.798	0.841	0.757	0.777	0.763
$T = 50000$	$\mathcal{C}_{\Re}$	0.800	0.845	0.857	0.742	0.846	0.865
	$\mathcal{C}_{\Im}$	0.999	0.980	0.966	0.998	0.981	0.974
	$\mathcal{C}_{\text{abs}}$	0.932	0.894	0.901	0.925	0.894	0.901
	$\mathcal{C}_{\text{joint}}$	0.799	0.830	0.827	0.740	0.828	0.839
$T = 100000$	$\mathcal{C}_{\Re}$	0.776	0.844	0.880	0.756	0.825	0.842
	$\mathcal{C}_{\Im}$	0.999	0.976	0.977	0.999	0.969	0.964
	$\mathcal{C}_{\text{abs}}$	0.930	0.883	0.915	0.914	0.879	0.884
	$\mathcal{C}_{\text{joint}}$	0.775	0.822	0.859	0.756	0.799	0.813
$T = 1000000$	$\mathcal{C}_{\Re}$	0.871	0.883	0.906	0.861	0.870	0.875
	$\mathcal{C}_{\Im}$	0.977	0.977	0.968	0.974	0.978	0.969
	$\mathcal{C}_{\text{abs}}$	0.940	0.924	0.916	0.913	0.915	0.909
	$\mathcal{C}_{\text{joint}}$	0.849	0.863	0.876	0.839	0.853	0.848

Table 4.10: Coverage results based on $u = 0.6, \delta = 0.75$ and $\delta_3 = 0.4$ together with $s = 0.6$ as well as $N = 1000$ and $N_{BS} = 2000$.

absolutely larger blocklengths to their disposal while using fewer observations.

At this point, we close this chapter dedicated to simulations and move on to independence testing as application to our findings. Another simulation study focusing on the later-presented testing procedure will be found in Chapter 7.

5 A Test of Independence Using a Weighted Characteristic Function-based Distance

This chapter addresses the question whether two locally stationary processes are index-wisely independent or not. To find an answer, we consult a weighted CF-distance inspired by the concept of distance covariance, which was introduced by Székely et al. (2007). Earlier, Kankainen and Ushakov (1998) already considered ECFs of functions of processes to test for independence using a weighted distance. Additionally to the actual weighted CF-distance, which will be defined in Section 5.1, we take an empirical version into account as well. The goal is to propose a test, which will be illustrated in the after next chapter by some simulations using bootstrap. To start our journey towards these simulations, we present the underlying concept and the real world theory here. Furthermore, this chapter closes with the proposition of the above-mentioned test. The transfer of the results into the bootstrap world, in turn, is subject of Chapter 6.

Returning to this chapter, we shine a light on the empirical weighted CF-distance for locally stationary processes based on the real world. Thereby, we proceed as follows: To begin the first section with, we present the essential concept of the weighted CF-distance including its empirical version. Then we adjust our assumptions and propose some assistant results. Moreover, we recommence the establishment of binding results, which we have already begun in Section 4.1 of the previous chapter. In this context, we will also accommodate some of our results of Chapter 2 to fit into the situation with $\varphi_X(u;\underline{s})$. Afterwards, we move on to the consistency of the empirical weighted CF-distance in the second section before ending with the corresponding asymptotic distribution in the last one.

5.1. Overview

Consider two random vectors $\underline{Y}$ and $\underline{Z}$ having their values in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Székely et al. (2007) defined a dependence measure based on the corresponding CFs and the joint one. The point of interest was the L_2-distance between joint and marginal CFs combined with a certain weight function. This allows for the measurement of non-linear dependence while requiring only mild moment conditions.

In Jentsch et al. (2020), this concept has been taken up and extended to fit the locally stationary framework surrounding the ECF-case. Contrary to this, we aim to apply only the concept of distance covariance, but not the actual framework itself to our setup. Our adaption will include a highly different choice of possible weight functions and lack the additional prefactor κ_T needed in the version in Jentsch et al. (2020). Another difference between Jentsch et al. (2020) and our consideration will be the focus solely on the lag $h = 0$. The reason behind this is the fact that we are interested in index-wise independence only.

Now we go more into detail. For this purpose, consider two locally stationary processes $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ and $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$ having their values in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, and meeting the requirements formulated in Assumption 1. Then, we have their companion processes $\left(\underline{\widetilde{Y}}_t(u)\right)_{t\in\mathbb{Z}}$ and $\left(\underline{\widetilde{Z}}_t(u)\right)_{t\in\mathbb{Z}}$, respectively, at rescaled time $u \in [0,1]$ as well. To this end, we define the weighted CF-distance by

$$\mathfrak{C}_{Y,Z}(u) := \int_{\mathbb{R}^p\times\mathbb{R}^q} \left|\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w} \tag{5.1}$$

with

$$\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) := Ee^{i\langle \underline{s}_1,\underline{\widetilde{Y}}_0(u)\rangle + i\langle \underline{s}_2,\underline{\widetilde{Z}}_0(u)\rangle}$$

as well as

$$\varphi_Y\left(u;\underline{s}_1\right) = Ee^{i\langle \underline{s}_1,\underline{\widetilde{Y}}_0(u)\rangle} \quad \text{and} \quad \varphi_Z\left(u;\underline{s}_2\right) = Ee^{i\langle \underline{s}_2,\underline{\widetilde{Z}}_0(u)\rangle}.$$

Moreover, it holds $d\mathfrak{w} = \mathfrak{w}\left(\underline{s}_1,\underline{s}_2\right)d\underline{s}_1 d\underline{s}_2$ for a positive weight function $\mathfrak{w}(\cdot,\cdot)$, which will be specified further in Assumption 10. We see in (5.1), CFs corresponding to the companion processes are used. Nevertheless, we aim to detect index-wise independence of the locally stationary process. Although this seems to be two different things, Lemma 2.3 in combination with the stationarity of the companion process allows us to consider independence of $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$ for fixed u as equivalent to t-wise independence of the locally stationary process if T tends to ∞. This is also the reason why we will assume this kind of independence for the companion process in our main results later on. The question of independence leads to the necessity of a positive weight function. In order to detect independence, the weighted CF-distance equals 0 if and only if $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$ are independent. Thus, the positivity of the weight function is crucial. Because the joint CF can be displayed as the product of the marginal CFs if and only if the belonging random vectors are independent, the integrand becomes 0 in the case of independence and is positive otherwise. The integral inherits this property.

At this point, we introduce the empirical weighted CF-distance as empirical pendant to equation (5.1) using observations $\underline{Y}_{1,T},\ldots,\underline{Y}_{T,T}$ and $\underline{Z}_{1,T},\ldots,\underline{Z}_{T,T}$. Analogously to (5.1), we define

$$\widehat{\mathfrak{C}}_{Y,Z}(u) := \int_{\mathbb{R}^p\times\mathbb{R}^q} \left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w} \tag{5.2}$$

with

$$\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) := \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\langle \underline{s}_1, \underline{Y}_{t,T}\rangle + i\langle \underline{s}_2, \underline{Z}_{t,T}\rangle}$$

plus

$$\widehat{\varphi}_{Y}\left(u;\underline{s}_1\right) = \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\langle \underline{s}_1, \underline{Y}_{t,T}\rangle}$$

and

$$\widehat{\varphi}_{Z}\left(u;\underline{s}_2\right) = \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\langle \underline{s}_2, \underline{Z}_{t,T}\rangle}.$$

Lastly, we have to specify the weight function $\mathfrak{w}$. In contrast to Székely et al. (2007), we will need $\mathfrak{w}$ to be integrable. To be more precise, the weight function has to satisfy the following assumption:

Assumption 10.
The weight function $\mathfrak{w}\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}_+$ *fulfills*

$$\int_{\mathbb{R}^p\times\mathbb{R}^q} \left(\left|\underline{s}_1\right|_1^{\frac{2+\delta}{2}} + \left|\underline{s}_2\right|_1^{\frac{2+\delta}{2}} + 1\right) \mathfrak{w}\left(\underline{s}_1,\underline{s}_2\right) d\underline{s}_1 d\underline{s}_2 < \infty.$$

Note that we will not extend our adaptions to the distance correlation as also defined in Székely et al. (2007). This would be problematic in combination with an integrable weight function as it is pointed out in said work.

Before we move on to the subsections containing the new findings, we draw attention to the fact that the centering performed in Jentsch et al. (2020) to establish their results is not necessary in our case because of the assumed independence of the companion processes. Moreover, the prefactor used there due to the non-integrable weight function becomes redundant in our setup.

5.2. A Consistency Result

In the previous chapters, we considered the origin of the first argument of the function, $\underline{s}$, to be a compact space. However, since the integrals in both equations (5.1) and (5.2) are not defined on such a space but on $\mathbb{R}^p\times\mathbb{R}^q$, we cannot apply the already established results any more. Therefore, we need to modify these findings to meet the new requirements. Moreover, we will adjust Assumption 2 as well. The adaptations will be made in such a way that $\underline{s}$ itself will be exposed. Thus, we do not have to work with upper bounds. The following assumption combines the modifications regarding Assumption 2 and previously imposed requisitions stated in Assumption 1 we maintain during this chapter.

Assumption 11 (Specialized Assumptions for Process and Functions f and g). *Suppose the validity of Assumption 1. Additionally, it holds the following:*

(i) In lieu of (2.2), condition (2.14) belonging to Assumption 4 is satisfied.

(ii) The function $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

- *is bounded by a finite constant* $C_{f,2} > 0$
- *and fulfills a Lipschitz condition in the following way: For* $\underline{s}, \underline{s}^\circ \in \mathbb{R}^d$ *and* $\mathbb{R}^d$*-valued random vectors* $\underline{X}, \underline{X}^\circ$*, it holds*

$$\left|f\left(\underline{s}, \underline{X}\right) - f\left(\underline{s}^\circ, \underline{X}^\circ\right)\right| \leq C_{Lip,2}\, g_s\left(\underline{s}, \underline{s}^\circ\right) \left|\underline{X} - \underline{X}^\circ\right|_1 + \left|\underline{s} - \underline{s}^\circ\right|_1 g\left(\underline{X}, \underline{X}^\circ\right) \quad (5.3)$$

  for a finite constant $C_{Lip,2} > 0$ *and functions* $g_s, g\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$.
- *The function* g_s *equals the* $|\cdot|_1$*-norm for* $\underline{s} = \underline{s}^\circ$*, that is*

$$g_s\left(\underline{s}, \underline{s}\right) := \left|\underline{s}\right|_1 . \quad (5.4)$$

Remark 5.1.

(i) The Lipschitz condition (5.3) applied to a function whose first argument belongs to a bounded range $[-S, S]^d$ equals the Lipschitz condition (2.11) introduced in Assumption 2.

(ii) Remembering the calculations made in Remark 2.5, we see that both the cosine and the sine function fulfill part (ii) of Assumption 11 with $C_{f,2} = 1$ in both cases. Moreover, equation (5.4) is satisfied as well. In conclusion, Assumption 11 and Assumption 2 combined with Assumption 1 are interchangeable if the sine and cosine function are considered and $\underline{s}$ lives on a compact space.

With the retrieved assumption above, we are now also enabled to modify some of our previously established results. Besides, we will intersperse some new findings as well. Said new results are purpose-built to fit the ECF-case, whereas the altered ones are partially of more general nature. Also, we tie in with Section 4.1 of the previous chapter and modify our main results of Chapter 2 in the bounded sense to the effect that they include $\varphi_X\left(u; \underline{s}\right)$. In this process, we proceed in the same order as in the previously mentioned chapter. The last result will be an adjusted FCLT, which plays an important role while proving the theorem approaching the asymptotic distribution of $\widehat{\mathfrak{C}}_{Y,Z}(u)$. Thus, we have to put a modified tightness as well as a restored CLT first. The CLT, in turn, requires an new version of the convergence of the covariance compared to Lemma 2.13. Hence, this will be the result we begin with:

Lemma 5.2 (Covariance II).
Let Assumption 9 be satisfied together with Assumption 11 for $k = 1$. Then, for $\underline{s}_1, \underline{s}_2 \in [-S, S]^d$ it holds

$$\begin{aligned}&\lim_{T\to\infty} \operatorname{Cov}\left(\Re\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}_1\right)\right), \Re\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}_2\right)\right)\right)\\ &\quad= \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \operatorname{Cov}\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h(u)\rangle\right)\right) := \sigma^2_{X,\Re}\left(u;\underline{s}_1,\underline{s}_2\right)\end{aligned}$$

and

$$\begin{aligned}&\lim_{T\to\infty} \operatorname{Cov}\left(\Im\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}_1\right)\right), \Im\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}_2\right)\right)\right)\\ &\quad= \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \operatorname{Cov}\left(\sin\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right), \sin\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h(u)\rangle\right)\right) := \sigma^2_{X,\Im}\left(u;\underline{s}_1,\underline{s}_2\right).\end{aligned}$$

Remark 5.3.

(i) In the same manner, covariances of combinations of real and imaginary part can be shown.

(ii) As a special case, we will denote the limit of the variance with $V_{X,\Re}\left(u;\underline{s}\right)$ and $V_{X,\Im}\left(u;\underline{s}\right)$, respectively. To be precise, we have

$$\begin{aligned}V_{X,\Re}\left(u;\underline{s}\right) &= \lim_{T\to\infty} \operatorname{Var}\left(\Re\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}\right)\right)\right)\\ &= \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h(u)\rangle\right)\right)\end{aligned}$$

and

$$\begin{aligned}V_{X,\Im}\left(u;\underline{s}\right) &= \lim_{T\to\infty} \operatorname{Var}\left(\Im\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}\right)\right)\right)\\ &= \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \operatorname{Cov}\left(\sin\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right), \sin\left(\langle \underline{s}, \underline{\widetilde{X}}_h(u)\rangle\right)\right).\end{aligned}$$

Now we can proceed with the modification of the CLT as stated below:

Theorem 5.4.
Suppose Assumption 9 is valid plus Assumption 11 for $k = 1$. In addition to that, let $b_T^3 T$ tend to 0 as T tends to ∞. Then, for $u \in [0,1]$ and $\underline{s} \in [-S,S]^d$ it holds

$$(b_T T)^{1/2} \begin{pmatrix} \Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right) \\ \Im\left(\widehat{\varphi}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\underline{0}, \mathbf{V}_{ECF}\left(u;\underline{s}\right)\right)$$

as T tends to ∞, where $\mathbf{V}_{ECF}\left(u;\underline{s}\right)$ is a covariance matrix with variance functions $V_{X,\Re}\left(u;\underline{s}\right)$ and $V_{X,\Im}\left(u;\underline{s}\right)$ on the main diagonal according to the second part of Remark 5.3. The counterdiagonal is of the same building type but using a covariance with both cosine and sine as arguments.

At this point, the adjusted version of the tightness is the only result lacking to modify the FCLT. Thus, we proceed to remedy this absence with the following lemma:

Lemma 5.5 (Tightness II).
Let Assumption 9 be fulfilled as well as 11 for $k = 1$. However, it holds

$$\sum_{j\in\mathbb{Z}} \frac{|j|^m}{l(j)} < \infty$$

for some

$$m > 1 + \frac{2}{\delta(1+\delta)}$$

instead of (2.14). Moreover, the innovations are equipped with finite $\left(\frac{2+\delta}{2}\right)$-th absolute moments and $b_T^3 T$ tends to 0 as $T \to \infty$. Then, we have

$$\lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|(b_T T)^{1/2} \Re\left(\left(\widehat{\varphi}_X\left(u;\underline{s}_1\right) - \varphi_X\left(u;\underline{s}_1\right)\right) - \left(\widehat{\varphi}_X\left(u;\underline{s}_2\right) - \varphi_X\left(u;\underline{s}_2\right)\right)\right)\right| > \lambda\right) = 0$$

and

$$\lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|(b_T T)^{1/2} \Im\left(\left(\widehat{\varphi}_X\left(u;\underline{s}_1\right) - \varphi_X\left(u;\underline{s}_1\right)\right) - \left(\widehat{\varphi}_X\left(u;\underline{s}_2\right) - \varphi_X\left(u;\underline{s}_2\right)\right)\right)\right| > \lambda\right) = 0$$

with $\lambda > 0$ and $\rho(\cdot,\cdot)$ denoting the $|\cdot|_1$-metric.

Finally, we are up to alter the FCLT as follows:

Theorem 5.6 (Functional Central Limit Theorem II).
Suppose the same assumptions as in Lemma 5.5 are satisfied. Additionally, let $b_T^3 T$ tend to 0 as $T \to \infty$. Then, it holds

$$\left((b_T T)^{1/2} \begin{pmatrix} \Re\left(\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right) \\ \Im\left(\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right) \end{pmatrix} \right)_{\underline{s}\in[-S,S]^d} \xrightarrow{d} \begin{pmatrix} G_{X,\Re}(\underline{s}) \\ G_{X,\Im}(\underline{s}) \end{pmatrix}_{\underline{s}\in[-S,S]^d}$$

as $T \to \infty$, where $\left(G_{X,\Re}(\underline{s})\right)_{\underline{s}\in[-S,S]^d}$ and $\left(G_{X,\Im}(\underline{s})\right)_{\underline{s}\in[-S,S]^d}$ are both centered Gaussian processes with continuous sample paths and covariance functions $\sigma^2_{X,\Re}(u;\underline{s},\underline{s}^\circ)$ and $\sigma^2_{X,\Im}(u;\underline{s},\underline{s}^\circ)$, respectively, as defined in Lemma 5.2. Here, convergence holds with respect to the uniform norm.

This new version of the FCLT terminates the modification of the already known findings specialized for the ECF-case, and we proceed with a newly designed lemma:

Lemma 5.7.
Let Assumption 9 be satisfied and also Assumption 11 for $k = 1$. Nevertheless, it holds

$$\sum_{j\in\mathbb{Z}} \frac{|j|^m}{l(j)} < \infty$$

for some

$$m > 1 + \frac{2}{\delta(1+\delta)}$$

in place of (2.14). Additionally, let the innovations have finite absolute moments of order $\frac{2+\delta}{2}$ and suppose $b_T^3 T \to 0$ as T tends to ∞. Then, it holds for any $u \in [0,1]$ and $\underline{s} \in [-S,S]^d$

$$\widehat{\varphi}_X(u;\underline{s}) \xrightarrow{d} \varphi_X(u;\underline{s})$$

as $T \to \infty$.

At this point, we start the modification of existing results in a more general way. Firstly, we deal with the difference between the companion process and its truncated version:

Lemma 5.8.
Let Assumption 11 be true for $k = 0$. Then, it holds for every $t \in \mathbb{Z}$, $M \in \mathbb{N}$, $\underline{s} \in \mathbb{R}^d$ and $u \in [0,1]$

$$\left\| f\left(\underline{s}, \underline{\widetilde{X}}_t(u)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}(u)\right) \right\|_1 \le C_{Lip,2} \left|\underline{s}\right|_1 \left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j|\ge M} \frac{B}{l(j)}.$$

Now, this lemma finds its use directly in the renewed version of the covariance bound stated below:

Lemma 5.9 (Covariance Bound II).
Suppose Assumption 11 is fulfilled for $k = 0$. Then, we have for all $u_1, u_2 \in [0, 1]$, $\underline{s} \in \mathbb{R}^d$ and every $h \in \mathbb{Z} \backslash \{0\}$

$$\left| \mathrm{Cov} \left(f \left(\underline{s}, \underline{\widetilde{X}}_0 (u_1) \right), f \left(\underline{s}, \underline{\widetilde{X}}_h (u_2) \right) \right) \right| \leq |\underline{s}|_1 \frac{C_{Cov,cs}}{|h|^{1+\tilde{\delta}}}$$

with $\tilde{\delta}$ as in Assumption 4 for some finite constant $C_{Cov,cs} > 0$, which is independent of $\underline{s}, h$ and both u_1 and u_2.

Remark 5.10.

(i) On account of stationarity, every pair of indices belonging to the companion process can be transformed to take on the desired form needed in Lemma 5.9.

(ii) Summability of the absolute values of the covariances is implicated by Lemma 5.9 as we have

$$\begin{aligned} &\sum_{h \in \mathbb{Z}} \left| \mathrm{Cov} \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0 (u_1) \right), f \left(\underline{s}_2, \underline{\widetilde{X}}_h (u_2) \right) \right) \right| \\ &\leq 2 \sum_{h \in \mathbb{N}} \left| \mathrm{Cov} \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0 (u_1) \right), f \left(\underline{s}_2, \underline{\widetilde{X}}_h (u_2) \right) \right) \right| \\ &\qquad + \left| \mathrm{Cov} \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0 (u_1) \right), f \left(\underline{s}_2, \underline{\widetilde{X}}_0 (u_2) \right) \right) \right|. \end{aligned}$$

The last summand, $\left| \mathrm{Cov} \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0 (u_1) \right), f \left(\underline{s}_2, \underline{\widetilde{X}}_0 (u_2) \right) \right) \right|$, is finite because of the boundedness of the function f. Hence, we obtain

$$\sum_{h \in \mathbb{Z}} \left| \mathrm{Cov} \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0 (u_1) \right), f \left(\underline{s}_2, \underline{\widetilde{X}}_h (u_2) \right) \right) \right| \leq 2 \, |\underline{s}|_1 \sum_{h \in \mathbb{N}} \frac{C_{Cov,cs}}{h^{1+\tilde{\delta}}} + C < \infty$$

as the sum over h portrays a geometric series.

Before we turn our attention back to some particular results for the ECF-case, we have to introduce some further notation. Often we need to establish a connection between the locally stationary process and its companion one but equipped with a certain argument. This can be done easily with the already shown results. Nevertheless, when the empirical version of the characteristic function comes into play, there exists no version involving the companion process. To facilitate the notation, we propose $\widehat{\widetilde{\varphi}} (u; \underline{s})$ to fill this gap. In greater detail, we consider

$$\widehat{\widetilde{\varphi}} (u; \underline{s}) := \frac{1}{b_T T} \sum_{t=1}^{T} K \left(\frac{t/T - u}{b_T} \right) e^{i \langle \underline{s}, \underline{\widetilde{X}}_t (u) \rangle} \tag{5.5}$$

for $\underline{s} \in \mathbb{R}^d$ and $u \in [0, 1]$. The next lemma's proof takes this filler up while quantifying the difference between $\widehat{\varphi}_X (u; \underline{s})$ and $\varphi_X (u; \underline{s})$.

Lemma 5.11.
Let Assumption 9 be valid and Assumption 11 for $k = 0$ as well. In addition to that, let the innovations have finite absolute moments of order $\frac{2+\delta}{2}$. Then, we have for every $u \in [0,1]$ and $\underline{s} \in \mathbb{R}^d$

$$E\left|\widehat{\varphi}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right|^2 = \left|\underline{s}\right|_1^{\frac{2+\delta}{2}} \mathcal{O}\left(b_T^{\frac{2+\delta}{2}}\right) + \left(\left|\underline{s}\right|_1 + 1\right)\mathcal{O}\left(\left(b_T T\right)^{-1}\right).$$

Here, the $\mathcal{O}$-terms are independent of the choice of $\underline{s}$ and u.

Remark 5.12.
Lemma 5.11 determines the exponent of b_T in the assumptions we will make for Theorem 5.14, which deals with the asymptotic distribution of $\widehat{\mathfrak{C}}_{Y,Z}(u)$, later on. Since we want this exponent to be as large as possible, we played on the highest order of finite absolutes moments available. This can be seen, for example, in (A.253). Instead of dividing the exponent the way we did, we could have chosen a split between 1 and 2δ followed by the bounding of the second factor. Because this lemma will find application in Theorem 5.14, where we will also assume the innovations to have finite absolute $\left(\frac{2+\delta}{2}\right)$-th moments for some $\delta \in (0,1)$ for various reasons, the demand of a higher order of finite absolute moments compared to Assumption 11 is justified.

Next, we aim to show the name-giving result of this section, that is consistency of $\widehat{\mathfrak{C}}_{Y,Z}(u)$, while using the results established above. As already mentioned earlier, there is no need for centering as done in Jentsch et al. (2020). This means, we show $\widehat{\mathfrak{C}}_{Y,Z}(u)$ is tending in probability to 0 as T tends to ∞. This determines the integrand which will be used to examine the asymptotic distribution in Section 5.3. This connectedness will play a role in the transfer to the bootstrap world in the following chapter as well.

Theorem 5.13 (Consistency of $\widehat{\mathfrak{C}}_{Y,Z}(u)$)**.**
Suppose Assumption 11 for $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ with $\underline{X}_{t,T} = \left(\underline{Y}_{t,T}', \underline{Z}_{t,T}'\right)'$ holds true for $k = 1$ and Assumptions 9 and 10 also. However, suppose the validity of

$$\sum_{j\in\mathbb{Z}} \frac{|j|^m}{l(j)} < \infty$$

for some

$$m > 1 + \frac{2}{\delta(1+\delta)}$$

instead of (2.14). Furthermore, let $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$ of the corresponding companion processes be independent for fixed $u \in [0,1]$. Additionally, assume the innovations have finite absolute moments of order $\frac{2+\delta}{2}$ and let $b_T^3 T$ tend to 0 as T tends to ∞. Then, we have

$$\widehat{\mathfrak{C}}_{Y,Z}(u) \xrightarrow{P} 0$$

as $T \to \infty$.

As already explained, the last result clears the way to the investigation of the asymptotic distribution of $\widehat{\mathfrak{C}}_{Y,Z}(u)$, which the following section lays the focus on.

5.3. An investigation of the Asymptotic Distribution

In this section, we concern ourselves with the examination of the asymptotic distribution of $\widehat{\mathfrak{C}}_{Y,Z}(u)$ both in the independent as well as the dependent case. The first will lead to a limit distribution, whereas the second engenders divergence.

Theorem 5.14 (Asymptotic Distribution of $\widehat{\mathfrak{C}}_{Y,Z}(u)$).
Let Assumption 11 for $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ with $\underline{X}_{t,T} := \left(\underline{Y}_{t,T}', \underline{Z}_{t,T}'\right)'$ be fulfilled for $k = 1$ as well as Assumptions 9 and 10. Nevertheless, let

$$\sum_{j\in\mathbb{Z}} \frac{|j|^m}{l(j)} < \infty$$

hold for some

$$m > 1 + \frac{2}{\delta(1+\delta)}$$

in lieu of (2.14). In addition, assume the representatives $\widetilde{\underline{Y}}_0(u)$ and $\widetilde{\underline{Z}}_0(u)$ of the corresponding companion processes to be independent for fixed $u \in [0,1]$. Besides, let the innovations have finite absolute moments of order $\frac{2+\delta}{2}$. Moreover, suppose $b_T^{\frac{4+\delta}{2}} T = \mathcal{O}(1)$. Then, it holds for any fixed $u \in [0,1]$

$$b_T T\, \widehat{\mathfrak{C}}_{Y,Z}(u) \xrightarrow{d} \int_{\mathbb{R}^p\times\mathbb{R}^q} \left|G\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2 d\mathfrak{w}$$

as T tends to ∞, where $\left(G\left(u;\underline{s}_1,\underline{s}_2\right)\right)_{(\underline{s}_1,\underline{s}_2)\in\mathbb{R}^p\times\mathbb{R}^q}$ is a centered Gaussian process with continuous sample paths and variance function

$$\begin{aligned}
&V_G\left(u;\underline{s}_1,\underline{s}_2\right)\\
&= V_{X,\Re}\left(u;\underline{s}\right) + V_{X,\Im}\left(u;\underline{s}\right) + \left|\varphi_Y\left(u;\underline{s}_1\right)\right|^2 V_{Z,\Re}\left(u;\underline{s}_2\right) + \left|\varphi_Y\left(u;\underline{s}_1\right)\right|^2 V_{Z,\Im}\left(u;\underline{s}_2\right)\\
&\qquad + \left|\varphi_Z\left(u;\underline{s}_2\right)\right|^2 V_{Y,\Re}\left(u;\underline{s}_1\right) + \left|\varphi_Z\left(u;\underline{s}_2\right)\right|^2 V_{Y,\Im}\left(u;\underline{s}_1\right)\\
&\qquad + 2\left(\Re\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right), G_{Z,\Re}\left(u;\underline{s}_2\right)\right)\right.\\
&\qquad - \Im\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right), G_{Z,\Im}\left(u;\underline{s}_2\right)\right)\\
&\qquad + \Re\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right), G_{Y,\Re}\left(u;\underline{s}_1\right)\right)\\
&\qquad - \Im\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right), G_{Y,\Im}\left(u;\underline{s}_1\right)\right)\\
&\qquad + \Im\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right), G_{Z,\Re}\left(u;\underline{s}_2\right)\right)\\
&\qquad + \Re\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right), G_{Z,\Im}\left(u;\underline{s}_2\right)\right)\\
&\qquad + \Im\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right), G_{Y,\Re}\left(u;\underline{s}_1\right)\right)\\
&\qquad \left. + \Re\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right), G_{Y,\Im}\left(u;\underline{s}_1\right)\right)\right).
\end{aligned}$$

Here, $\left(G_{X,\Re}\left(u;\underline{s}\right)\right)_{\underline{s}\in\mathbb{R}^{p+q}}$ *and* $\left(G_{X,\Im}\left(u;\underline{s}\right)\right)_{\underline{s}\in\mathbb{R}^{p+q}}$ *denote the centered Gaussian processes originating from the application of Theorem 5.6 with regard to the real and imaginary part of the ECF belonging to the process* $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$*. The appurtenant covariance functions are* $\sigma^2_{X,\Re}\left(u;\underline{s},\underline{s}^\circ\right)$ *and* $\sigma^2_{X,\Im}\left(u;\underline{s},\underline{s}^\circ\right)$*, respectively, as expressed in Lemma 5.2. The same is in effect for the processes* $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ *and* $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$*.*

Remark 5.15.
Theorem 5.14 imposes an additional constraint to the bandwidth. Together with Assumption 9, we have

$$b_T^2 T \underset{T\to\infty}{\longrightarrow} \infty \quad \text{and} \quad b_T^{\frac{4+\delta}{2}} T = \mathcal{O}(1).$$

Thus, an additional $\delta/2$ in the exponent of b_T decides whether the product diverges or not. This is for small δ, which means fewer finite absolute moments, rather restrictive.

After the successful establishment of a limit distribution in the independent case, we will now investigate the behavior of $\widehat{\mathfrak{C}}_{Y,Z}(u)$ while both processes are in a relationship of dependence. This takes place in the following lemma:

Lemma 5.16.
Suppose Assumption 11 for $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$*, where* $\underline{X}_{t,T} := \left(\underline{Y}'_{t,T}, \underline{Z}'_{t,T}\right)'$*, holds true for* $k = 1$ *as well as Assumptions 9 and 10. However, let*

$$\sum_{j\in\mathbb{Z}} \frac{|j|^m}{l(j)} < \infty$$

hold true for some

$$m > 1 + \frac{2}{\delta(1+\delta)}$$

in place of (2.14). Besides, let $\widetilde{\underline{Y}}_0(u)$ *and* $\widetilde{\underline{Z}}_0(u)$ *belonging to the corresponding companion processes be dependent for fixed* $u \in [0,1]$*. Additionally, assume the innovations have finite* $\left(\frac{2+\delta}{2}\right)$*-th absolute moments and let* $b_T^3 T \to 0$ *as* T *tends to* ∞*. Then, it holds for any fixed* $u \in [0,1]$

$$b_T T\, \widehat{\mathfrak{C}}_{Y,Z}(u) \overset{P}{\longrightarrow} \infty$$

as T *tends to* ∞*.*

With this last lemma, we close the purely theoretical part and move on to the testing procedure.

5.4. A Testing Procedure for Independence

This section addresses the aforementioned test for independence and a starting point, where to make use of bootstrap in this case. To this end, consider the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$, where $\underline{X}_{t,T} := \left(\underline{Y}'_{t,T}, \underline{Z}'_{t,T}\right)'$ is composed of the processes $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ and $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$.

A testing procedure for independence using weighted CF-distance is based on the asymptotic distribution of the empirical weighted CF-distance $\widehat{\mathfrak{C}}_{Y,Z}(u)$. As we have seen in Theorem 5.14 and Lemma 5.16, with underlying dependence, $\widehat{\mathfrak{C}}_{Y,Z}(u)$ tends to ∞ while $T \to \infty$, whereas independence results in an actual limit distribution. Thus, we would like to decide upon the value of $\widehat{\mathfrak{C}}_{Y,Z}(u)$ whether the null hypothesis of $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ and $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$ being t-wisely independent can be rejected or not. In order to do that, we have to classify $\widehat{\mathfrak{C}}_{Y,Z}(u)$ for a certain finite value of T as we cannot simulate a truly infinite sample size. Said classification should tell us whether we opt for underlying independence or reject the null hypothesis. The use of a threshold is not far to seek in this case. Thus, a useful quantile of the limit distribution established in Theorem 5.14 would do the trick. As Lemma 5.16 attests a divergence to infinity if we have dependent processes, we are only in need for an upper bound contrary to the simulation study in Chapter 4. Then, given $T \to \infty$ and $\mathfrak{p} \in (0,1)$, the percentage of a false rejection of independence would be $\mathfrak{p} \cdot 100$ leading to a significance level of $\mathfrak{p}$. As we work with observations, we would have to estimate the quantile in question and hence the variance of the limit distribution. However, this would be very elaborate and computation-intensive as well. To solve this problem, an empirical bootstrap quantile as seen in Chapter 4 would come in handy. This proceeding would require at least the existence of a bootstrap pendant to Theorem 5.14 with the same limit distribution and hence the same variance. That is the reason why we close the real world consideration at this point and move on to the bootstrap world.

6 A Bootstrap Counterpart for the Weighted Characteristic Function-based Distance

In analogy to Chapter 3, the object of this chapter is to transfer the results established in the previous one to the bootstrap world on the basis of Algorithm 3.1. We start by presenting the notation and with the introduction of some further assumptions. Then we work on some preparatory results. These results can be divided into two groups. First, we establish those specifically made for the ECF-case. Thereafter, we look at general ones. The need for the latter comes from the fact that we do not longer consider the origin of $\underline{s}, \underline{s}_1$ and $\underline{s}_2$ to be a compact space as already described in section 5.2. Comparable to the real world findings, we have to modify the corresponding bootstrap results as well. Additionally, some convergence results, which are tailor-made for the proofs of the two theorems in this chapter, can be found in the corresponding parts of Appendix A. Eventually, we are able to establish the bootstrap versions of the consistency and the asymptotic distribution of $\widehat{\mathfrak{C}}_{Y,Z}(u)$ presented in Sections 5.2 and 5.3, respectively.

6.1. Overview

The first step of transferring the real world findings into the bootstrap world is the definition of a bootstrap counterpart to the empirical weighted CF-distance. Based on (5.2), we consider

$$\begin{aligned}\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) = \int_{\mathbb{R}^p\times\mathbb{R}^q} \Big| &\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) \\ &+E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\, E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\Big|^2 \, d\mathfrak{w} \quad (6.1)\end{aligned}$$

with the help of

$$\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) := \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\langle \underline{s}_1, \underline{Y}^{\star}_{t,T}\rangle + i\langle \underline{s}_2, \underline{Z}^{\star}_{t,T}\rangle}$$

as well as

$$\widehat{\varphi}_Y^\star(u;\underline{s}_1) := \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\langle \underline{s}_1, \underline{Y}_{t,T}^\star\rangle}$$

and

$$\widehat{\varphi}_Z^\star(u;\underline{s}_2) := \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\langle \underline{s}_2, \underline{Z}_{t,T}^\star\rangle}.$$

The differences between (5.2) and (6.1) are reasoned by the fact that we cannot relinquish the centering any longer. Thus, we incorporated it directly into definition (6.1). Since the equality between the joint CF and the product of the marginal CFs under independence is not transferable to the bootstrap expectation of the bootstrap version of the ECF, we obtain additional terms to deal with.

Before we continue with the additional results tailor-made for the ECF-case, we have to modify our assumptions in the following way:

Assumption 12.
In addition to the fulfilment of Assumptions 6, 9 and 11, it holds the following:

(i) The innovations are equipped with finite absolute moments of order $\frac{2+\delta}{2}$.

(ii) The sequence of bandwidths $(b_T)_{T\in\mathbb{N}}$ satisfies $b_T^3 T = o(1)$.

Remark 6.1. The restrictive constraint addressed in Remark 5.15 looses its need because of the modified form of $\widehat{\mathfrak{C}}_{Y,Z}^\star(u)$ compared to $\widehat{\mathfrak{C}}_{Y,Z}(u)$.

6.2. A Bootstrap Consistency Result

After being acquainted with the new notation, we proceed with the preliminary work, which allows us to state a bootstrap version of Theorem 5.13 at the end of this section.

Up next, we have two results specially tailored for the bootstrap version of the ECF. The first interrelates the bootstrap ECF with the CF in two ways:

Lemma 6.2.
Let Assumption 12 be fulfilled for $k=0$. Then, it holds for every $\underline{s}\in\mathbb{R}^d$ and $u\in[0,1]$

(i)

$$\left|\widehat{\varphi}_X^\star(u;\underline{s}) - \varphi_X(u;\underline{s})\right|^2 = \left(|\underline{s}|_1 + 1\right)\mathcal{O}_{P^\star}(b_T)$$

and

(ii)

$$\left|E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right|^2=\left(\left|\underline{s}\right|_1+1\right)\mathcal{O}_P\left(b_T\right).$$

The occurring $\mathcal{O}_{P^\star}$- and $\mathcal{O}_P$-term are both independent of $\underline{s}$ and u, respectively.

Remark 6.3.
If $\underline{s}$ is element of $[-S,S]^d$, we use the Lipschitz condition stated in Assumption 2 instead of (5.3). Then, the factor $\left(\left|\underline{s}\right|_1+1\right)$ can be bounded by a constant and, thus, incorporated in the $\mathcal{O}_P$- and $\mathcal{O}_P^\star$-terms. Therefore, we obtain also

(i)

$$\sup_{\underline{s}\in[-S,S]^d}\left|\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right|^2=\mathcal{O}_{P^\star}\left(b_T\right)$$

as well as

(ii)

$$\sup_{\underline{s}\in[-S,S]^d}\left|E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right|^2=\mathcal{O}_P\left(b_T\right)$$

with $\mathcal{O}_{P^\star}$- and $\mathcal{O}_P$-terms independent of u.

The second of the abovementioned findings is a convergence result dealing with the difference between the bootstrap ECF and its bootstrap expectation. It reads as follows:

Lemma 6.4.
Suppose Assumption 12 holds true for $k=1$. Then, we have for any $\underline{s}\in[-S,S]^d$ and $u\in[0,1]$

$$\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)\xrightarrow{P^\star}0$$

in P-probability and uniformly in $\underline{s}$.

As already mentioned before, the integration area in (6.1) does not form a compact space. Thus, we have to adapt some results of Chapter 2 as well as of Section 3.5 to meet the new situation. At it, most of the proofs rely heavily on the structure belonging to the ones suitable for the former versions of the results. We start with the examination of the transition from the locally stationary process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ to the companion one while forming the second argument of the function f.

Lemma 6.5.
Let Assumption 11 holds true for $k=0$. Then, we have for any $\underline{s}\in\mathbb{R}^d$

$$\sup_{1\leq t\leq T}\left\|f\left(\underline{s},\underline{X}_{t,T}\right)-f\left(\underline{s},\widetilde{\underline{X}}_t\left(\frac{t}{T}\right)\right)\right\|_1=\left|\underline{s}\right|_1\frac{C_{B',2}}{T}.$$

for some positive constant $C_{B',2}<\infty$ independently of $\underline{s}$.

The next point of interest is the bound for the covariance dealing with products of f. The following lemma shines a light on this:

Lemma 6.6 (Product Covariance Bound III).
Suppose Assumption 11 for $k = 1$ is satisfied.

(i) Then, for all $u \in [0,1]$, $\underline{s} \in \mathbb{R}^d$ and $t_1 \in \mathbb{N}$ and $t_2, r \in \mathbb{N}_0$ with $t_1 > t_2$, we have

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \leq |\underline{s}|_1 \frac{C_{Cov,2i,b'}}{(t_1 - t_2)^{1+\tilde{\delta}}}.$$

(ii) Moreover, for all $u \in [0,1]$, $\underline{s} \in \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{N}$ satisfying $t_1 < t_2$, it holds

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \leq \begin{cases} \frac{C_{Cov,2ii,b'}}{(t_2-t_1)^{1+\tilde{\delta}}}, & t_1 \leq \frac{t_2}{2}, \\ \frac{C_{Cov,2ii,b'}}{t_1^{1+\tilde{\delta}}}, & t_1 > \frac{t_2}{2}. \end{cases}$$

The inequalities of parts (i) and (ii) hold both for $\tilde{\delta}$ as in Assumption 4, some positive constants $C_{Cov,2i,b'}, C_{Cov,2ii,b'} < \infty$ and with $\bar{f}$ signifying the centered version of f introduced in (2.15). *Here, said constants do not depend either on u and $\underline{s}$ or on t_1, t_2 and r.*

Remark 6.7.
As Lemmata 3.4 and 3.14 already did, Lemma 6.6 also indicates the summability of absolute values ot this covariance type. This can, exemplarily, be seen for the summation index $v := t_1 - t_2 > 0$ for both fixed t_2 and r. In this case, we have

$$\begin{aligned} &\sum_{v \in \mathbb{N}} \left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+t_2}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\ &\leq |\underline{s}|_1 \sum_{v \in \mathbb{N}} \frac{C_{Cov,2i,b'}}{v^{1+\tilde{\delta}}} \\ &< \infty \end{aligned}$$

fox fixed $\underline{s}$.

Regarding the already established results of this subsection, we notice that they all have a merely preparatory character. The next lemma, in turn, benefits more heavily from those preliminary findings. To be precise, we want to transform the bootstrap covariance into the one appurtenant to the real world while paying close attention to the first argument of f, namely $\underline{s}$.

Lemma 6.8.
Let Assumptions 6 and 11 for $k = 1$ be valid. Then, for all indices $t_1, t_2 \in 1, \ldots, T$ and $u \in [0,1]$ as well as $\underline{s} \in \mathbb{R}^d$ it holds

$$\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left(\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \right) \right.$$
$$\left. \cdot \left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \right) \right)$$
$$= \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \right) + \left(|\underline{s}|_1 + 1\right) \mathcal{O}_P\left((TD_T)^{-1}\right). \tag{6.2}$$

The above-mentioned $\mathcal{O}_P$-term does not depend on t_1, t_2, u or $\underline{s}$.

In Sections 3.4 and 3.6, we used good sets to prove the tightness findings. We will adopt this concept to prove both the consistency and the asymptotic distribution result regarding $\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$ later in this section. Prior to this, we have to identify suitable good sets. To this end, we begin by the examination of a difference akin to those already investigated in Lemma 6.2. This result, in turn, will form a decent starting position for the determination of a convenient good set. As in the mentioned sections before, we shift the actual identification of this and all following good sets into the Appendix A or, more precisely, into Subsection A.5.1.

Returning to the aforementioned difference result, we consider the following lemma:

Lemma 6.9.
Suppose Assumption 12 holds true for $k = 0$. Then, we have

$$E^{\star} \left|\widehat{\varphi}^{\star}_X(u; \underline{s}) - E^{\star}\widehat{\varphi}^{\star}_X(u; \underline{s})\right|^2 = \frac{1}{b_T T}\left(|\underline{s}|_1 + 1\right)\left(\mathcal{O}(1) + \mathcal{O}_P\left(\frac{L_T}{TD_T}\right)\right)$$

for all $\underline{s} \in \mathbb{R}^d$ and $u \in [0,1]$, which do not influence the $\mathcal{O}$- or the $\mathcal{O}_P$-term.

At this point, we are able to move on to the bootstrap version of the consistency result in Theorem 5.13. As already explained, the equality between the joint CF and the product of the marginal CFs under independence cannot be transferred to the bootstrap expectation of the bootstrap version of the ECF. This makes the need for presupposed independence obsolete. Thus, both the consistency result as well as Theorem 6.11 hold whether the locally stationary or the companion processes are independent or not. The concomitant absence of divergence allows for the use of the bootstrap results in our testing procedure later on.

Theorem 6.10 (Consistency of $\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$)**.**
Suppose Assumption 12 for $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ with $\underline{X}_{t,T} = \left(\underline{Y}'_{t,T}, \underline{Z}'_{t,T}\right)'$ holds true for $k = 1$ and Assumption 10 also. Then, we have

$$\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) \xrightarrow{P^{\star}} 0$$

in P-probability as $T \to \infty$.

Now we are ready to show the behavior of the asymptotic distribution of $\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$ in the next section.

6.3. An Investigation of the Bootstrap Asymptotic Distribution

As we have argued yet, there is no need to distinguish between independence or dependence of the two processes $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ and $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$ and their companion ones. Therefore, there is no explicit bootstrap analogue to Lemma 5.16. Instead, both cases are covered in the following theorem:

Theorem 6.11 (Asymptotic Distribution of $\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$).
Suppose Assumption 12 for $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ with $\underline{X}_{t,T} := \left(\underline{Y}'_{t,T}, \underline{Z}'_{t,T}\right)'$ is true for $k = 1$ as well as Assumption 10. Then, it holds for any fixed $u \in [0,1]$

$$b_T T\, \widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) \xrightarrow{d} \int_{\mathbb{R}^p \times \mathbb{R}^q} \left| G\left(u; \underline{s}_1, \underline{s}_2\right)\right|^2 d\mathfrak{w}$$

in P-probability, where $\left(G\left(u; \underline{s}_1, \underline{s}_2\right)\right)_{(\underline{s}_1, \underline{s}_2) \in \mathbb{R}^p \times \mathbb{R}^q}$ is a centered Gaussian process with continuous sample path and variance function $V_G\left(u; \underline{s}_1, \underline{s}_2\right)$ as stated in Theorem 5.14.

Remark 6.12.
With the last result, we see that even while it covers both the independent and the dependent case, we still got the desired bootstrap-induced limit distribution addressed in subsection 5.4. It does not matter if the underlying observations originate from dependent or independent processes because the empirical bootstrap quantile will always mimic the one belonging to the independent real world case. Thus, the newly established quantile provides the threshold we were looking for. This enables us to proceed with our testing idea from earlier in the following chapter.

7 A Simulation Study for Testing of Independence

This chapter takes the testing idea up again, which was proposed in Section 5.4, and combines it with the bootstrap results established in the previous chapter. Moreover, we show the performance of this testing procedure via various simulations. As we consider independence as the null hypothesis, the results we obtain are the rates indicating the non-rejection rate of said hypothesis. Thus, this chapter is divided into two sections: The first addresses the simulation setup, whereas in the second, we show the simulation results and discuss them. In order to diminish computational cost, an alternative computation method can be found in Subsection B.2.1 of Appendix B. This method will be used throughout the simulations.

7.1. The Simulation Setup

In most parts, we use nearly the same setup for our simulation as in Chapter 4 only adjusted to the scenario of a composed process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ with $\underline{X}_{t,T} := (Y_{t,T}, Z_{t,T})'$. To be precise, we stick to the ECF-case and the underlying α-stable distribution for the processes $(Y_{t,T})_{t=1}^{T}$ and $(Z_{t,T})_{t=1}^{T}$. In this scenario, this translates to

$$Y_{t,T} = 0.9 \sin\left(2\pi \frac{t}{T}\right) Y_{t-1,T} + \varepsilon_{Y,t}$$

for $t = 2, \dots, T$ plus

$$Y_{1,T} = 0.9 \sin\left(2\pi \frac{1}{T}\right) \varepsilon_{Y,0} + \varepsilon_{Y,1}$$

for $t = 1$ and

$$Z_{t,T} = 0.9 \sin\left(2\pi \frac{t}{T}\right) Z_{t-1,T} + \varepsilon_{Z,t}$$

for $t = 2, \dots, T$ together with

$$Z_{1,T} = 0.9 \sin\left(2\pi \frac{1}{T}\right) \varepsilon_{Z,0} + \varepsilon_{Z,1}$$

for $t = 1$. Here, both innovation series $(\varepsilon_{Y,t})_{t\in\mathbb{N}_0}$ and $(\varepsilon_{Z,t})_{t\in\mathbb{N}_0}$ are i.i.d. Furthermore, we assume both series to have the same marginal distribution. Clearly, if the belonging innovation processes are independent, the processes $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ and $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$ are t-wisely independent as well due to the way the locally stationary processes are built. Moreover, we do not change the parameters belonging to the innovations' marginal distribution, that is $\mu = 0, \alpha = 1.5, \beta = 0$ and $\gamma = 0.5$, which were introduced firstly in equation (4.3). Nevertheless, due to the tightened assumptions concerning the bandwidth, as thematized in Remark 5.15, we have to alter at least our choice of δ_3. However, this time we put the cart before the horse and start with the deliberation about manageable sample sizes regarding memory capacity. It turns out that $T = 5000$ is the largest sample size our computers are able to work with. Besides, we set our goal for a blocklength of at least $L_T = 2$ to preserve the benefits the block bootstrap method brings along. In addition to that, we would like to investigate the testing procedure using $L_T = 3$ for a part of the examined sample sizes as we have seen in Chapter 4 that larger blocklengths tend to result in better coverage rates. Considering all given constraints, be it by the assumptions in the former chapters or by our choice, we obtain the following values as possible candidates:

$$\delta = 0.91,\ \delta_1 = 0.14,\ \delta_2 = 0.8 \quad \text{and} \quad \delta_3 = 0.41.$$

Moreover, due to this parameter choice the kernel function filters the endpoints out again comparable to the scenario in Section 4.2. Regarding the parameter u, we adopt at first both choices of Chapter 4, albeit there is the reasoned guess that the procedure using $u = 0.6$ will perform better due to the better coverage results regarding the imaginary part. The reason therefore is that in the testing procedure, real and imaginary part are combined as the ECFs are used as a whole. Concerning the sample size, we begin with $T = 500$ and increase the value stepwise. The same applies to the numbers of repetitions N and N_{BS} starting with 100 and 200, respectively. These numbers are far below our choices in Chapter 4, but we have to defer to the available computational capacity.

Throughout the whole simulation study, we choose the significance level $\mathfrak{p} = 0.05$. In other words, we aim for a non-rejection rate of 0.95 under the null hypothesis.

7.2. The Simulation Results

This section contains the results of our simulations. Furthermore, we present our interpretation of said results.

To begin with, we conduct simulations with smaller values for T, N and N_{BS} to get an overview about the result's behavior for different values of u. To be precise, we start with the pairs $(100; 200), (250; 500)$ and $(500; 1000)$ for N and N_{BS} while considering $T \in \{500, 1000\}$. The simulations for $T = 5000$, which will be the maximal number of observations we will perform our simulations with, are very time-consuming. Therefore, we refrain from including the maximal value for the number of observations in our overview.

Tables 7.1 and 7.2 show the resulting non-rejection rates for $u = 0.4$ and $u = 0.6$, respectively. For $T = 1000$, we notice that the non-rejection rate increases monotonously, whereas this is not the case for $T = 500$. Though, this can be explained by the fact that higher variation is normal for smaller T and smaller repetition numbers. Overall, the values for $T = 1000$ surpass the ones for smaller T only for the last pair of repetition numbers. However, this does not portend a general impairment of the non-rejection rates for larger sample sizes. Instead, the small numbers can be held to account for this once more. Besides, indicative for this explanation is the fact that the maximum for the non-rejection rate can be found for $T = 1000, N = 500$ and $N_{BS} = 1000$. Being 0.676, the archived rate leaves room to meet the favored 0.95.

$u = 0.4$	$N = 100$, $N_{BS} = 200$	$N = 250$, $N_{BS} = 500$	$N = 500$, $N_{BS} = 1000$
$T = 500$	0.670	0.664	0.630
$T = 1000$	0.650	0.656	0.676

Table 7.1: Non-rejection rates based on independent innovation series for $u = 0.4$.

Now we turn our attention to Table 7.2, that is the one based on $u = 0.6$. This time, we notice higher values for $T = 1000$ compared to $T = 500$ in all three cases. Another difference is that the rates for $T = 1000$ do not grow monotonously. But, again, the reason therefore could be found in the small observation number combined with little repetitions. Remembering the numbers we used in Chapter 4, even $T = 1000$ is a small one. Returning to Table 7.2, the overall maximum is 0.802 for $T = 1000, N = 500$ and $N_{BS} = 1000$, which is closer at 0.95 as the maximal rate for $u = 0.4$. In general, comparing all results in Tables 7.1 and 7.2 it is clearly visible that the results using $u = 0.6$ are better, even with the variation due to little observations. Note that these conclusions also fit in well with those drawn in Section 4.3.

$u = 0.6$	$N = 100$, $N_{BS} = 200$	$N = 250$, $N_{BS} = 500$	$N = 500$, $N_{BS} = 1000$
$T = 500$	0.780	0.748	0.764
$T = 1000$	0.800	0.764	0.802

Table 7.2: Non-rejection rates based on independent innovation series for $u = 0.6$.

To confirm our choice regarding u, we take another sample size into account, namely $T = 2559$. This odd number has its origins in the targeted blocklength of $L_T = 3$ since it is the smallest T, for which this particular blocklength is possible while maintaining our parameter choice. Because augmenting the number of observation does only have an impact on the blocklength if a certain threshold is overrun, we settle for the smallest T possible to keep the runtime moderate. In Table 7.3, the results are written down for 100 real world repetitions and 200 bootstrap ones. The achieved rates are higher than the ones for smaller T. In addition to that, we see the difference between $u = 0.4$ and $u = 0.6$ clearly, despite the fact that the achieved 0.86 might become apparent as a lucky strike on the upper scale in comparison to the results obtained otherwise. Thus, we continue our simulations with the latter.

$T = 2559$	$N = 100, N_{BS} = 200$
$u = 0.4$	0.720
$u = 0.6$	0.860

Table 7.3: Non-rejection rates based on independent innovation series for a sample size $T = 2559$.

Because of the remarkable difference between the as yet achieved rate at the maximum and 0.95, a question which suggests itself is whether the choice of the value for α is a unfortunate one. To find an answer, we consider $\alpha = 2$ for the next few of simulations. This choice results in a Gaussian distribution, which performs more favorably in most cases if possible. Our choices for δ_1, δ_2 and δ_3 stay the same because they result from the choice for δ. An enhancement of the value for δ would not automatically eventuate in higher-performing values for δ_1, δ_2 and δ_3 since its influence is working in opposite directions regarding these values. The optimization of the value chosen for δ could be another jumping-off point for rate improvement, but we keep on the already begun path. Table 7.4 contains the results for $N = 200$ and $N_{BS} = 500$ for the already used observation numbers to maintain comparability. We see that, even with augmented bootstrap repetitions, the results are not better than the ones for $\alpha = 1.5$. Au contraire, for $T = 2559$, yet the non-rejection rate for $N = 100, N_{BS} = 200$ and $\alpha = 1.5$, that is 0.86, exceeds the one for $\alpha = 2$ using smaller repetition numbers, which counts for 0.795. In conclusion, the choice for α does not seem to be responsible for not reaching any rate near 0.95.

$u = 0.6, \alpha = 2$	$N = 200, N_{BS} = 500$
$T = 500$	0.840
$T = 1000$	0.805
$T = 2559$	0.795

Table 7.4: Non-rejection rates based on independent innovations following a Gaussian distribution for $u = 0.6$.

Even by redoubling the number of bootstrap repetitions, we only achieve a non-rejection rate of 0.805 for $T = 2559$ with the new value for α as it can be seen in Table 7.5.

$u = 0.6, \alpha = 2$	$N = 200, N_{BS} = 1000$
$T = 2559$	0.805

Table 7.5: Non-rejection rate based on independent innovation series for $u = 0.4$ with $N_{BS} = 1000$.

Therefore, we return to $\alpha = 1.5$. Moreover, we augment the number of observation to maximal capacity, that is $T = 5000$. Using both $N_{BS} = 500$ and $N_{BS} = 1000$ while leaving N to be equal to 200, we obtain the results shown in Table 7.6. We observe nearly the same results in both cases with non-rejection rates of 0.835 and 0.825, respectively. Moreover, we can identify a slight upswing in the rates for increasing observation number. Based on this, we hope for further improvement of the non-rejection rates by a stronger increasement of the sample size. However, this is a job for computers possessing a higher performance than the ones we have at our disposal.

$u = 0.6$	$N = 200$	
	$N_{BS} = 500$	$N_{BS} = 1000$
$T = 5000$	0.835	0.825

Table 7.6: Non-rejection rates based on independent innovation series for $u = 0.6$ and $T = 5000$ with increasing bootstrap repetitions.

As we cannot see any improvement by the enhancement of the bootstrap repetition number, we go over to play around with N instead of N_{BS}. Therefore, we retrieve the non-rejection rate for $T = 5000$ and $N_{BS} = 200$ at $N = 200, 400$ and 750. In Table 7.7, we do not see any change in the results, which stay at 0.825 the whole time. This substantiates the guess that possible improvement could most likely be achieved by augmenting the observation number.

$u = 0.6$	$N_{BS} = 1000$		
	$N = 200$	$N = 400$	$N = 750$
$T = 5000$	0.825	0.825	0.825

Table 7.7: Non-rejection rates based on independent innovation series for $u = 0.6$ and $T = 5000$ with increasing numbers of outer loops.

However, after presuming underlying independence of the two locally stationary processes in question it is left to look at the dependent counterpart and to verify whether the dependence can be detected by the testing procedure or not. To this end, we consider three different scenarios. The first deals with one of the directest dependence structures as possible, namely by using the same innovation process for both $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ and $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$. In addition to that, we address two other dependence structures of the innovations, which are inspired by Bakirov et al. (2006) and Székely et al. (2007). The first assumes $(\varepsilon_{Y,t})_{t\in\mathbb{N}_0}$ to follow a standard Gaussian distribution. Based thereon, the other innovation series forms as follows:

$$\varepsilon_{Z,t} := \log\left(\varepsilon_{Y,t}^2\right)$$

for all $t \in \mathbb{N}_0$. The last dependence scenario we are considering is the most complicated out of the three. Here, $(\varepsilon_{Y,t})_{t\in\mathbb{N}_0}$ has, again, a standard Gaussian distribution. In addition, we introduce an auxiliary innovation process $(\varepsilon_t)_{t\in\mathbb{N}_0}$, which follows a standard Gaussian distribution as well. Besides, $(\varepsilon_t)_{t\in\mathbb{N}_0}$ and $(\varepsilon_{Y,t})_{t\in\mathbb{N}_0}$ are supposed to be independent. Then, the lacking innovation series is defined as

$$\varepsilon_{Z,t} := \varepsilon_{Y,t}\,\varepsilon_t$$

for all $t \in \mathbb{N}_0$. Subsequently, we will denote the last two scenarios by *log dependence* and *product dependence*, respectively. We redo the simulations already conducted for the independent case to see what happens using dependent innovation series. In order to maintain a reasonable runtime, we opt for $N = 200$ and $N_{BS} = 500$. However, we perform the simulations for different sample sizes, to wit $T = 1000, 2559$ and 5000. The resulting non-rejection rates can be found in Table 7.8. For comparison, the ones based on independence are listed as well. First, it stands out that in the simulations using the same distribution for both innovation series, the dependence has been detected in every case. Moreover, the more indirect the dependence becomes, the higher are the non-rejection rates for every choice of T. Regarding the log dependence and the product dependence one by one, we notice a decrease in the rates for with augmenting numbers of observation. This fuels the theory of the need for larger sample size for better results. Overall, the non-rejection rate for log dependence can be brought down to 0 as minimum, whereas the one corresponding to the product dependence has its lowest value at 0.025.

	$u = 0.6$	$N = 200, N_{BS} = 500$
$T = 1000$	independent	0.775
	same distribution	0.000
	log dependence	0.020
	product dependence	0.365
$T = 2559$	independent	0.815
	same distribution	0.000
	log dependence	0.005
	product dependence	0.155
$T = 5000$	independent	0.835
	same distribution	0.000
	log dependence	0.000
	product dependence	0.025

Table 7.8: Non-rejection rates based on independent and dependent innovation series for $u = 0.6$.

Concluding, the obtained results of these simulations consort with the results of Chapter 4 in regard to the impact the used sample size has. However, we obtained better results for comparably small T in this study. This might be caused by the consideration of combinations of both real and imaginary part of the ECF in the test statistic, which allows for compensatory effects. Another similarity is the choice of u to obtain better results.

This closes the second simulation-dedicated chapter and. Therewith, the main part of this thesis.

8 Conclusions

Now our journey has come to an end. Therefore, it is time to recapitulate what we have seen and done. We started in the real world with a general overview of the setup we were going to work with. In this context, we established some basic results, which were helpful on our whole way. Moreover, as the initial motivation was the generalization of the FCLT in Jentsch et al. (2020), a passing joint work, we carved out the starting points for said generalization. Since it became clear that stronger assumptions would be needed for unbounded functions, the idea of a two-pronged approach focusing either on an unbounded or bounded function f was born. Then, we approached the desired FCLT with detours for covariance results, a CLT and a tightness result. After establishing the FCLT, the end of Chapter 2 was reached, and thus the first stage of our journey was done. In conclusion, we were able to transfer the results in Jentsch et al. (2020) without especially strong assumptions to a more general setup. The division in two versions referring to the boundedness of f allowed for adjusting the assumptions to prevent unnecessary strong ones. Moreover, we have managed to relax the summability condition for the coefficients $(A_{t,T}(j))_{j\in\mathbb{Z}}$ imposed in Jentsch et al. (2020) for the bounded case.

Then we moved on to the second world in Chapter 3. After a short overview, we took the bootstrap algorithm of Dowla et al. (2013) and modified it together with the corresponding assumptions to suit our purpose. With the help of the altered algorithm, we began with the transfer of the results established in Chapter 2 to the bootstrap world. In contrast to said chapter, the differences between the bounded and unbounded case were more severe, which resulted in a sharp separation between distinct sections. In Sections 3.3 and 3.4, we stuck to the unbounded case, which made additional assumptions concerning the maximal order of finite absolute moments necessary. Although Chapter 2 acted as a guideline, we were in need for additional results due to the bootstrap scenario. However, some of these auxiliary results took place in the real world and were used for the transition between the two worlds. Subsequently, we repeated the stages of Sections 3.3 and 3.4 in the following two addressing the bounded case regarding the function f. Because some of the yet established results could be reused or became obsolete, these sections were not as expansive as the former. Recapitulatory, we could establish bootstrap analogues to the FCLT in both cases regarding f, which included all findings leading the way to said theorem as well. Except for altered assumptions in terms of absolute finite moments and summability conditions, we were not obligated to impose further requisites with respect to the starting scenario to show theses results.

After two theoretical chapters, the next one included a practical aspect. In Chapter 4, we picked up on the ECF-case, which was already known from Jentsch et al. (2020). After some additional results to fit the kernel function perfectly, we shone a light on the class of α-stable distributions, which would supply the underlying distribution for our simulation study. Said study was conducted directly afterwards and followed by an interpretation of the results. In general, our procedure worked well. Nevertheless, we identified several ways to improve the obtained coverage results and pointed out where other improvement potential could be found.

In the following chapter, we went back to a more theoretical part, which, however, concluded in a testing procedure for independence using ECFs. Inspired by the distance covariance application in Jentsch et al. (2020), we presented a weighted CF-distance, where we used an integrable weight function. Additionally, some assumptions had to be also modified. As before in Chapter 4, we were in need for some additional results tailor-made for the ECF-case along with modifications of already established results due to an unbounded parameter space for the first argument of f. For this modifications, we stayed in the general framework to maintain the possibility of wider application. This led to the proposal of a testing procedure at the end of Chapter 5. At it, we indicated the implementation benefits from having a bootstrap analogue to the already established results, which determined the subject of the next chapter.

In Chapter 6, we considered the results of the previous one and transferred them to the bootstrap world with all given peculiarities. After having presented the bootstrap analogue for the real world weighted CF-distance, we altered some assumptions. Then, the transfer of the findings took place in the same manner as in Chapter 3. This cleared the way to conduct a simulation study to apply our testing procedure.

Finally, we were able to observe the way our testing procedure worked in practice. Our setup stayed basically the same as in Chapter 4. However, we availed ourselves of the observations made with regard to the parameter choice. Illustrated by several tables, we obtained results which tended in the right direction, but also aroused justified guess of being able to be improved by more powerful computers. On the other hand, for underlying dependence the testing procedure worked well even with limited computational resources. On balance, all results induced us to see our testing procedure as a success, which would be worth further testing with better performing computers.

With this last thoughts, we bring our adventure through the worlds to completion.

A | Proofs

This first chapter of the appendix addresses itself entirely to the proofs of the results established in the preceding chapters with the assistance of some auxiliary results, whose proofs can be found here as well. The structure of this chapter mimics the order of the results in the main part. In other words, each section approaches a certain chapter's findings intermitted by ancillary results if needed. To be even more precise, the single sections from the main part including provable results find their counterparts in form of subsections here.

A.1. Proofs Belonging to Chapter 2

This section breaks the first ground on our proof journey. We return to the beginning of this thesis and start with the proofs belonging to the results stated in Section 2.1.

A.1.1. Proofs of Section 2.1

The first lemma to prove is Lemma 2.3, and its proof reads as follows:

Proof of Lemma 2.3. Since the change of the exponent in the summability condition has no impact on the proof of Lemma 2.3, we will follow the lines of Jentsch et al. (2020) in general while proving both parts. However, for a better understanding of the meaning of Assumption 1, we will go more into detail in some places.

(i) Because both the locally stationary and the companion process rely on the same innovations, we can rewrite the difference using (2.1) and (2.7) as follows:

$$\left\| \underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right\|_1 = \left\| \sum_{j\in\mathbb{Z}} \left(A_{t,T}(j) - A\left(\frac{t}{T}, j\right)\right) \underline{\varepsilon}_{t-j}\right\|_1. \tag{A.1}$$

Making use of the submultiplicativity of the $|\cdot|_1$-norm and the fact that the innovations are i.i.d., we obtain

$$\left\| \underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right\|_1 \leq \sum_{j\in\mathbb{Z}} \left| A_{t,T}(j) - A\left(\frac{t}{T}, j\right)\right|_1 \|\underline{\varepsilon}_0\|_1 \tag{A.2}$$

instead of (A.1). Due to equation (2.5) of Assumption 1, we can bound the right-hand side of (A.2) by

$$T^{-1} \left\|\underline{\varepsilon}_0\right\|_1 \sum_{j\in\mathbb{Z}} \frac{B}{l(j)} = \frac{C_B}{T}.$$

This finishes the first part.

(ii) With help of representation (2.7) and the same arguments as in the previous part, the difference in question can be bounded via

$$\begin{aligned}\left\|\underline{\widetilde{X}}_t\left(u_1\right) - \underline{\widetilde{X}}_t\left(u_2\right)\right\|_1 &\leq \left|\underline{\mu}\left(u_1\right) - \underline{\mu}\left(u_2\right)\right|_1 + \left\|\underline{\varepsilon}_0\right\|_1 \sum_{j\in\mathbb{Z}} \left|A\left(u_1, j\right) - A\left(u_2, j\right)\right|_1 \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{A.3}$$

Since part (iii) of Assumption 1 requires each component of the mean function $\underline{\mu}$ to be continuously differentiable, each of those components is Lipschitz continuous as well. Therefore, we get for the first term

$$\mathrm{I} = \sum_{r=1}^{d} \left|\mu_r\left(u_1\right) - \mu_r\left(u_2\right)\right| \leq C_\mu\, d \left|u_1 - u_2\right|,$$

where C_μ denotes the maximum of the d Lipschitz constants belonging to the components of the mean function $\underline{\mu}$. Turning to the second term of (A.3), we use a Taylor expansion for each entry $a^{(p,q)}(k,j)$ corresponding to the matrix $A\left(u,j\right)$. In combination with equation (2.4) of Assumption 1, we obtain

$$\begin{aligned}\mathrm{II} &= \left\|\underline{\varepsilon}_0\right\|_1 \sum_{j\in\mathbb{Z}} \max_{q=1,\ldots,d} \sum_{p=1}^{d} \left|A\left(u_1, j\right) - A\left(u_2, j\right)\right|_1 \\ &\leq \left\|\underline{\varepsilon}_0\right\|_1 \sum_{j\in\mathbb{Z}} \max_{q=1,\ldots,d} \sum_{p=1}^{d} \left|a^{(p,q)}\left(u_1, j\right) - a^{(p,q)}\left(u_2, j\right)\right| \\ &= \left\|\underline{\varepsilon}_0\right\|_1 \sum_{j\in\mathbb{Z}} \max_{q=1,\ldots,d} \sum_{p=1}^{d} \left|\left(\frac{\partial a^{(p,q)}(k,j)}{\partial q}\Big|_{k=\xi_{p,q,j}}\right) \cdot \left(u_1 - u_2\right)\right| \\ &\leq \left\|\underline{\varepsilon}_0\right\|_1 d \sum_{j\in\mathbb{Z}} \frac{B}{l(j)} \left|u_1 - u_2\right|\end{aligned}$$

using appropriate $\xi_{p,q,j}$ lying between $a^{(p,q)}\left(u_1, j\right)$ and $a^{(p,q)}\left(u_2, j\right)$. Joining both results, we get

$$\left\|\underline{\widetilde{X}}_t\left(u_1\right) - \underline{\widetilde{X}}_t\left(u_2\right)\right\|_1 \leq d\left(C_\mu + \left\|\underline{\varepsilon}_0\right\|_1 \sum_{j\in\mathbb{Z}} \frac{B}{l(j)}\right) \left|u_1 - u_2\right| = C_{\bar{B}} \left|u_1 - u_2\right|_1,$$

which brings part (ii) and thereby the proof to completion.

□

We continue with a series of short proofs dealing with the basic findings of the first section of Chapter 2. The next proof belongs to Lemma 2.6.

Proof of Lemma 2.6. Using the Lipschitz condition (2.11) and part (i) of Lemma 2.3, it holds

$$\sup_{\substack{1\leq t\leq T\\ \underline{s}\in[-S,S]^d}} \left\| f\left(\underline{s},\underline{X}_{t,T}\right) - f\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right\|_1 \leq C_{Lip} \sup_{1\leq t\leq T} \left\|\underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right\|_1 \leq \frac{C_{B'}}{T}$$

with $C_{B'} := C_{Lip}\, C_B$ where C_B originates from the first part of Lemma 2.3. □

Subsequently, we turn to the proof appurtenant to Lemma 2.8.

Proof of Lemma 2.8. First, we make use of the Lipschitz condition (2.11) and obtain

$$\begin{aligned} E&\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\left(f\left(\underline{s},\underline{X}_{t,T}\right) - f\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right)\right|\right) \\ &\leq E\left(\sup_{\underline{s}\in[-S,S]^d}\sum_{t=1}^{T} w_{t,T}\, C_{Lip}\left|\underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right|_1\right). \end{aligned} \tag{A.4}$$

Thus, we see that the supremum becomes superfluous. Therefore, neglecting the constant we continue with

$$E\left(\sum_{t=1}^{T} w_{t,T}\left|\underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right|_1\right) \tag{A.5}$$

in preference to (A.4). Using Lemma 2.6, (A.5), in turn, can be bounded by

$$\sum_{t=1}^{T} w_{t,T}\left\|\underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right\|_1 \leq C_w\, d_T^{-1/2}\,\frac{C_{B'}}{T} \leq C_w\, C_{B'}\, d_T^{1/2}.$$

In consequence, setting $C_{sup} := C_{Lip}\, C_w\, C_{B'}$ brings the proof to completion. □

The second to last proof of this subsection belongs to Lemma 2.9.

Proof of Lemma 2.9. With use of the Lipschitz condition (2.11), it holds

$$\sup_{\underline{s}\in[-S,S]^d}\left\| f\left(\underline{s},\underline{\widetilde{X}}_t(u)\right) - f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}(u)\right)\right\|_1 \leq C_{Lip}\left\|\underline{\widetilde{X}}_t(u) - \underline{\widetilde{X}}_t^{(M)}(u)\right\|_1.$$

Now we can investigate the difference. As the mean function is not affected by the truncation, we get

$$\left\| \underline{\widetilde{X}}_t(u) - \underline{\widetilde{X}}_t^{(M)}(u) \right\|_1 = \left\| \sum_{|j| \geq M} A(u,j)\, \underline{\varepsilon}_{t-j} \right\|_1 .$$

Since the innovations are i.i.d., we can extract them from the sum and get using equation (2.3)

$$\left\| \sum_{|j| \geq M} A(u,j)\, \underline{\varepsilon}_{t-j} \right\|_1 \leq \left\| \underline{\varepsilon}_0 \right\|_1 \sum_{|j| \geq M} |A(u,j)|_1 \leq \left\| \underline{\varepsilon}_0 \right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)}.$$

Recapitulatory, we obtain

$$\sup_{\underline{s} \in [-S,S]^d} \left\| f\left(\underline{s}_1, \underline{\widetilde{X}}_t(u)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t^{(M)}(u)\right) \right\|_1 \leq C_{Lip} \left\| \underline{\varepsilon}_0 \right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)},$$

which is the result we aimed to show. □

Finally, we close this subsection with the proof of Lemma 2.10, which is some more complex than the previous ones.

Proof of Lemma 2.10. The first step will be to split the product with the help of Hoelder's inequality. By choosing the Hoelder exponents wisely, we are able to make use of the $(2+\delta)$-th finite absolute moments the function f has. Hence, we obtain

$$\begin{aligned}
&\left\| \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right) f\left(\underline{s}_2, \underline{\widetilde{X}}_{t_2}(u_2)\right) \right\|_1 \\
&\leq \left\| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right\|_{\frac{2+\delta}{1+\delta}} \left\| f\left(\underline{s}_2, \underline{\widetilde{X}}_{t_2}(u_2)\right) \right\|_{2+\delta} \\
&\leq C \left(E \left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right|^{\frac{2+\delta}{1+\delta}} \right)^{\frac{1+\delta}{2+\delta}} .
\end{aligned} \tag{A.6}$$

Now we continue with the remaining expectation of the difference. Once again, we use Hoelder's inequality. But before being able to do that, we need to divide the exponent in order to get a product to apply the said inequality to. Then, we will choose the Hoelder exponents while having in mind that we aim to bring Lemma 2.9 to bear. In so doing, we get

$$\begin{aligned}
&\Bigg(E \Bigg(\left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right|^{\frac{\delta(2+\delta)}{(1+\delta)^2}} \\
&\qquad \cdot \left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right|^{\frac{2+\delta}{(1+\delta)^2}} \Bigg) \Bigg)^{\frac{1+\delta}{2+\delta}}
\end{aligned}$$

$$\leq \left(E \left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right| \right)^{\frac{\delta}{1+\delta}} \cdot \left(E \left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right|^{\frac{2+\delta}{(1+\delta)^2}\cdot(1+\delta)^2} \right)^{\frac{1}{(1+\delta)(2+\delta)}} \tag{A.7}$$

instead of the expectation in (A.6). At this point, we can use Lemma 2.9 to bound the first expectation in (A.7) and obtain therewith

$$\left(\left\|\underline{\varepsilon}_{t_1}\right\|_1 d \sum_{|j|\geq M} \frac{B}{l(j)} \right)^{\frac{\delta}{1+\delta}} \left(E \left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right|^{2+\delta} \right)^{\frac{1}{(1+\delta)(2+\delta)}}.$$

Since in the remaining expectation we have to deal with finite absolute moments of order $2+\delta$ the function f has, we can finally deduce

$$\begin{aligned}\left\| \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right) f\left(\underline{s}_2, \underline{\widetilde{X}}_{t_2}(u_2)\right) \right\|_1 \\ \leq C_{DP} \left(\left\|\underline{\varepsilon}_{t_1}\right\|_1 d \sum_{|j|\geq M} \frac{B}{l(j)} \right)^{\frac{\delta}{1+\delta}}\end{aligned}$$

for some finite constant $C_{DP} > 0$, which finishes the proof. □

A.1.2. Proofs of Section 2.2

After the preliminary results are proven, we move on the findings of Section 2.2. In this subsection, the proofs gain on length and complexity compared to the ones shown before. The first pertains to Lemma 2.11.

Proof of Lemma 2.11. W.l.o.g. consider $h \geq 1$. Our first step will be to insert the truncated process $\left(\underline{\widetilde{X}}_t^{(M)}\right)_{t\in\mathbb{Z}}$ as described in equation (2.13) for $M := \lceil h/2 \rceil$ into this bound by adding and subtracting a mixed covariance at the same time. Consequently, we get

$$\begin{aligned}
&\left| \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right) \right) \right| \\
&= \Bigg| \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right) \right) - \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right) \right) \\
&\qquad + \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right) \right) \\
&\qquad - \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h^{(M)}(u_2)\right) \right) \Bigg| \\
&\leq \left| \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right) \right) \right| \\
&\qquad + \left| \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h^{(M)}(u_2)\right) \right) \right|
\end{aligned}$$

$=: \mathrm{I} + \mathrm{II}.$

As it can clearly be seen, terms I and II have the same building type. Hence, we focus on the first one. Now we treat the cases (a) and (b) differently.

(a) We start by splitting term I up further. To this end, we bound said term from above by

$$E\left(\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right)\right|\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right|\right) \\ + E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right)\right| E\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right|$$

$=: \mathrm{Ia} + \mathrm{Ib}.$

Continuing with the investigation of subterm Ib, we make use of Assumption 2 to bound the second factor as it holds

$$E\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right| \leq C$$

for some $C < \infty$. Thus, the expectation of the difference is left, which, in turn, can be bounded by Lemma 2.9. Therefore, we obtain

$$\mathrm{Ib} \leq C \left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)}.$$

Then, we can bring our attention to Ia. Using Lemma 2.10, we can bound the said subterm by

$$C_{DP} \left(\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}}.$$

Consequently, we can establish an upper bound for I via

$$\mathrm{I} \leq C \left(\left(\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)}\right).$$

For the reason explained earlier, by following the exact same arguments we get

$$\mathrm{II} \leq C \left(\left(\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j| \geq M} \frac{B}{l(j)}\right)$$

as well. Combining these two results, it holds for the absolute value of the covariance

$$
\begin{aligned}
&\left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(u_1\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(u_2\right)\right)\right)\right| \\
&\leq C\left(\left(\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j|\geq M} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j|\geq M} \frac{B}{l(j)}\right) \\
&\qquad\qquad\qquad + C\left(\left(\sum_{|j|\geq\lceil h/2\rceil} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \sum_{|j|\geq\lceil h/2\rceil} \frac{B}{l(j)}\right) \\
&\leq C\left(\left(\sum_{|j|\geq\lceil h/2\rceil} \frac{|j|^{\frac{(1+\tilde{\delta})(1+\delta)}{\delta}} B}{|j|^{\frac{(1+\tilde{\delta})(1+\delta)}{\delta}} l(j)}\right)^{\frac{\delta}{1+\delta}} + \sum_{|j|\geq\lceil h/2\rceil} \frac{j^2 B}{j^2 l(j)}\right) \\
&\leq C\left(\left(\frac{1}{h^{\frac{(1+\tilde{\delta})(1+\delta)}{\delta}}} \sum_{|j|\geq\lceil h/2\rceil} \frac{|j|^{\frac{(1+\tilde{\delta})(1+\delta)}{\delta}} B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \frac{1}{h^2} \sum_{|j|\geq\lceil h/2\rceil} \frac{j^2 B}{l(j)}\right) \\
&\leq C_{ca}\left(h^{-\frac{(1+\tilde{\delta})(1+\delta)}{\delta}\cdot\frac{\delta}{1+\delta}} + h^{-2}\right) \\
&\leq \frac{C_{ca}}{h^{1+\tilde{\delta}}} \qquad\qquad (\mathrm{A}.8)
\end{aligned}
$$

for some constant $0 < C_{ca} < \infty$. The last step is possible since we have $\tilde{\delta} \in (0,1)$ and hence $1+\tilde{\delta} < 2$.

(b) As in part (a), we have a closer look at term I and get

$$
\begin{aligned}
&E\left(\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(u_1\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(u_1\right)\right)\right|\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(u_2\right)\right)\right|\right) \\
&\qquad\qquad + E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(u_1\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(u_1\right)\right)\right| E\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(u_2\right)\right)\right| \\
&\leq 2\, C_f\, E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(u_1\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(u_1\right)\right)\right| \\
&\leq C\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j|\geq M} \frac{B}{l(j)}
\end{aligned}
$$

as an upper bound with use of Lemma 5.8. Again, by repeating the same steps for term II we get

$$
\mathrm{II} \leq C\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j|\geq M} \frac{B}{l(j)}
$$

and therewith

$$
\begin{aligned}
\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(u_1\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(u_2\right)\right)\right)\right| &\le C\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{|j|\ge M} \frac{B}{l(j)} \\
&\le C \sum_{|j|\ge\lceil h/2\rceil} \frac{|j|^{1+\tilde{\delta}} B}{|j|^{1+\tilde{\delta}} l(j)} \\
&\le C \frac{1}{h^{1+\tilde{\delta}}} \sum_{|j|\ge\lceil h/2\rceil} \frac{|j|^{1+\tilde{\delta}} B}{l(j)} \\
&\le \frac{C_{cb}}{h^{1+\tilde{\delta}}} \qquad \text{(A.9)}
\end{aligned}
$$

for some positive constant $C_{cb} < \infty$.

Closing, we bring parts (a) and (b) together, and by setting $C_{Cov} := \max\left\{C_{ca}, C_{cb}\right\}$ with C_{ca} and C_{cb} being the constants in (A.8) and (A.9), respectively, the proof is completed. □

After having shown the covariance bound, we move on to covariance convergence result presented in Lemma 2.13.

Proof of Lemma 2.13. On the right-hand side of (2.16), the lag between the arguments of the covariance is stated implicitly by the difference between t_1 and t_2. So, the first step will be to rewrite the right-hand side of (2.16) in order to create a sum dedicated to the lag denoted explicitly by h. Hence, we get

$$
\begin{aligned}
&\lim_{T\to\infty} \sum_{t_1,t_2=1}^{T} \operatorname{Cov}\left(w_{t_1,T}\, f\left(\underline{s}_1, \underline{X}_{t_1,T}\right), w_{t_2,T}\, f\left(\underline{s}_2, \underline{X}_{t_2,T}\right)\right) \\
&= \lim_{T\to\infty} \sum_{t=1}^{T} \sum_{h=-(t-1)}^{T-t} \operatorname{Cov}\left(w_{t,T}\, f\left(\underline{s}_1, \underline{X}_{t,T}\right), w_{t+h,T}\, f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right) \\
&= \lim_{T\to\infty} \sum_{h=-(T-1)}^{T-1} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right). \qquad \text{(A.10)}
\end{aligned}
$$

Next, we want to insert the limit into the sum over h. Therefore, we show the existence of an integrable upper bound for the inner sum. Then, we can apply Lebesgue's theorem to change the places of the outer sum and the limit. Using the same arguments as in Lemma 2.11, we obtain for the covariance

$$
\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\right| \le \frac{C}{|h|^{1+\tilde{\delta}}}
$$

for every $h \in \mathbb{Z}\backslash\{0\}$. For $h = 0$, we get using the Cauchy-Schwarz inequality

$$
\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t,T}\right)\right)\right| \le \left(\operatorname{Var}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right)\right) \operatorname{Var}\left(f\left(\underline{s}_2, \underline{X}_{t,T}\right)\right)\right)^{1/2} \le C_1,
$$

since f has finite $(2+\delta)$-th absolute moments. Now we can bound the inner sum of (A.10) by

$$\begin{aligned}&\left|\sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T}\ \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\right| \\ &\leq \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T}\, \frac{C}{|h|^{1+\tilde{\delta}}} \\ &\leq \frac{C}{|h|^{1+\tilde{\delta}}}\end{aligned}$$

for $h \in \mathbb{Z}\backslash\{0\}$ and by

$$\left|\sum_{t=1}^{T} w_{t,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t,T}\right)\right)\right| \leq C_1 \sum_{t=1}^{T} w_{t,T}^2 \leq C_1$$

in the case $h = 0$. Since it holds

$$2\sum_{h\in\mathbb{N}} \frac{C}{|h|^{1+\tilde{\delta}}} + C_1 < \infty,$$

we have found an integrable bound for the inner sum of (A.10) and can make use of Lebesgue's theorem. Hence, we obtain

$$\begin{aligned}&\lim_{T\to\infty} \sum_{h=-(T-1)}^{T-1} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T}\ \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right) \\ &= \sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \left(\mathbb{1}_{\{|h|<(T-1)\}} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T}\ \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\right) \\ &= \sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T}\ \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right). \qquad \text{(A.11)}\end{aligned}$$

Because the companion process takes the place of the locally stationary process on the left-hand side of (2.16), our next step will be to transform (A.11) into the limit of a weighted sum of covariances of $\left(\widetilde{\underline{X}}_t(u)\right)$. Therefore, we focus on the covariance in (A.11), which we can rewrite in the following way:

$$
\begin{aligned}
&\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\\
&= \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\\
&\qquad + \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{t+h}\left(\frac{t+h}{T}\right)\right)\right)\\
&\qquad\qquad + \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_{t+h}\left(\frac{t+h}{T}\right)\right)\right)\\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Since III is the expression we are looking for, we show the asymptotic negligibility of the first two terms. Starting with term I, we get using Lemma 2.6

$$
\begin{aligned}
\mathrm{I} &= \Bigg| E\left(\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right) f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\\
&\qquad - E\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right) E f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\Bigg|\\
&\le E\left(\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| \left|f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right|\right)\\
&\qquad + E\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| E\left|f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right|\\
&\le E\left(\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| \left|f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right|\right) + \frac{C}{T}.
\end{aligned}
$$

Inspired by the proof of Lemma 2.11, we obtain for the first summand from above with double use of Hoelder's inequality

$$
\begin{aligned}
&E\left(\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| \cdot \left|f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right|\right)\\
&= \left\|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right\|_{\frac{2+\delta}{1+\delta}} \left\|f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right\|_{2+\delta}\\
&\le C\left(E\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right|^{\frac{2+\delta}{1+\delta}}\right)^{\frac{1+\delta}{2+\delta}}\\
&= C\left(E\left(\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right|^{\frac{\delta(2+\delta)}{(1+\delta)^2}}\right.\right.\\
&\qquad\qquad\left.\left. \cdot \left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right|^{\frac{2+\delta}{(1+\delta)^2}}\right)\right)^{\frac{1+\delta}{2+\delta}}
\end{aligned}
$$

$$\leq C \left(\left(E \left| f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right|\right)^{\frac{\delta}{1+\delta}} \cdot \left(E \left| f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right|^{\frac{2+\delta}{(1+\delta)^2}\cdot(1+\delta)^2}\right)^{\frac{1}{(1+\delta)(2+\delta)}}\right)$$

$$\leq \frac{C}{T^{\frac{\delta}{1+\delta}}},$$

where the last step is the application of Lemma 2.6 once again. Since we have $T^{-1} \leq T^{-\frac{\delta}{1+\delta}}$ as it holds $-\frac{\delta}{1+\delta} > -1$, it follows $\mathrm{I} = \mathcal{O}\left(T^{-\frac{\delta}{1+\delta}}\right)$. Analogously, we have $\mathrm{II} = \mathcal{O}\left(T^{-\frac{\delta}{1+\delta}}\right)$. Altogether, it holds

$$\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right) = \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_{t+h}\left(\frac{t+h}{T}\right)\right)\right) + \mathcal{O}\left(T^{-\frac{\delta}{1+\delta}}\right).$$

Thus, we can write (A.11) as follows:

$$\begin{aligned}
&\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\\
&= \sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \cdot \left(\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_{t+h}\left(\frac{t+h}{T}\right)\right)\right) + \mathcal{O}\left(T^{-\frac{\delta}{1+\delta}}\right)\right)\\
&= \sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_{t+h}\left(\frac{t+h}{T}\right)\right)\right).
\end{aligned} \tag{A.12}$$

Due to stationarity, we are allowed to change the indices of the companion process to work with

$$\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \tag{A.13}$$

instead of (A.12). In addition to that, we want to change the bounds of the inner sum to eliminate the minimum and maximum in order to get

$$\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \tag{A.14}$$

to pursue the proof with. The first step in doing so is to ensure equation (A.14) is well-defined. To this end, we introduce some notational adaptions and set

$$\underline{\widetilde{X}}_t(z) = \begin{cases} \underline{\widetilde{X}}_t(1), & z > 1, \\ \underline{\widetilde{X}}_t(0), & z < 0, \end{cases}$$

for all $t \in \mathbb{Z}$ and

$$w_{t,T} = \begin{cases} w_{T,T}, & t > T, \\ w_{1,T}, & t < 1. \end{cases}$$

Next, we have to verify that the difference between (A.13) and (A.14) can be bounded adequately so that it vanishes as T tends to infinity. Note that there only exists a difference if h is unequal to 0. Therefore, we assume this is the case for the following calculations concerning the questionable difference. The aforementioned bound behaves in the following way:

$$\begin{aligned}
&\Bigg| \sum_{h \in \mathbb{Z}} \lim_{T \to \infty} \Bigg(\sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \\
&\qquad - \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right)\Bigg)\Bigg| \\
&\leq \sum_{h \in \mathbb{Z}} \lim_{T \to \infty} \Bigg| \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \\
&\qquad - \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right)\Bigg|.
\end{aligned}$$

We put the outer sum on hold for a moment and concentrate on the inner part, which can be rewritten and bounded as

$$\begin{aligned}
&\Bigg| \sum_{t=1}^{T-h} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \mathbb{1}_{\{h \in [1,T-1]\}} \\
&\quad + \sum_{t=1-h}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \mathbb{1}_{\{h \in [-(T-1),-1]\}} \\
&\qquad - \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right)\Bigg| \\
&= \Bigg| - \sum_{t=T-h+1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \mathbb{1}_{\{h \in [1,T-1]\}} \\
&\qquad - \sum_{t=1}^{-h} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \mathbb{1}_{\{h \in [-(T-1),-1]\}}\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \sum_{t=T-h+1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \mathbb{1}_{\{h\in[1,T-1]\}} \right| \\
&\qquad + \left| \sum_{t=1}^{-h} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \mathbb{1}_{\{h\in[-(T-1),-1]\}} \right| .
\end{aligned} \tag{A.15}
$$

Again, we make use of Lemma 2.11 to bound (A.15) by

$$
\frac{C_{Cov}}{|h|^{1+\tilde{\delta}}} \left(\sum_{t=T-h+1}^{T} w_{t,T}\, w_{t+h,T}\, \mathbb{1}_{\{h\in[1,T-1]\}} + \sum_{t=1}^{-h} w_{t,T}\, w_{t+h,T}\, \mathbb{1}_{\{h\in[-(T-1),-1]\}} \right) = \frac{C}{|h|^{\tilde{\delta}}}\, d_T,
$$

which converges to 0 as $T \to \infty$ for fixed h. The next step will be to change the argument of $\underline{\widetilde{X}}_h(\cdot)$ in (A.14) to eliminate h in the numerator. We can rewrite the covariance by adding a zero term to obtain

$$
\begin{aligned}
&\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \\
&= \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\
&\qquad + \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right).
\end{aligned} \tag{A.16}
$$

So the next step will be to bound the latter covariance term to show its asymptotic negligibility. This means, we can replace the original covariance with the one with the same arguments of the companion process. To bound the latter summand the way we aim for, we need to investigate $\left|E\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right|$ further. With the help of the Lipschitz condition 2.11 in Assumption 2 and the second part of Lemma 2.3, it holds

$$
\begin{aligned}
&\left|E\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq C_1 \left\| \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right) - \underline{\widetilde{X}}_h\left(\frac{t}{T}\right) \right\|_1 \\
&\leq C_1 \frac{|h|}{T},
\end{aligned}
$$

which converges for fixed h to 0 as $T \to \infty$. Now we go back to the second covariance term of (A.16) and establish an upper bound for it. We have

$$
\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq E\left(\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right|\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\right) \\
&\qquad\qquad + E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right| E\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right| \\
&= E\left(\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right|\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\right) + \mathcal{O}\left(\frac{|h|}{T}\right)
\end{aligned}
$$

as we have seen in the calculation before. Next, remembering the use of Hoelder's inequality in the proof of Lemma 2.11, we obtain

$$
\begin{aligned}
&E\left(\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right|\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\right) \\
&\leq \left\|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right\|_{2+\delta}\left\|f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right\|_{\frac{2+\delta}{1+\delta}} \\
&\leq C_2\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}}.
\end{aligned}
$$

Since it holds $|h|/T < 1$ and therefore

$$
\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}} \geq \frac{|h|}{T},
$$

we get

$$
\begin{aligned}
&\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\
&\leq C_2\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}} + C_1\frac{|h|}{T} \\
&= \mathcal{O}\left(\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}}\right)
\end{aligned}
$$

altogether. For fixed h, this converges to 0 for $T \to \infty$. At this point, we return to (A.14). Having the results from above in mind, we get

$$
\begin{aligned}
&\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right) \\
&= \sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \\
&\qquad\qquad \cdot \left(\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) + \mathcal{O}\left(\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{h \in \mathbb{Z}} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \,\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\
&= \sum_{h=-\infty}^{\infty} V_h\left(\underline{s}_1, \underline{s}_2\right).
\end{aligned}
$$

So the first part of the lemma is shown. To complete the proof, we have to show the existence of the limit. Using Lemma 2.11 once again, we demonstrate that the upper bound of the term in question is bounded. In doing so, we obtain

$$
\begin{aligned}
&\left| \sum_{h=-\infty}^{\infty} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \,\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq 2 \sum_{h=1}^{\infty} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \frac{C_{Cov}}{h^{1+\tilde{\delta}}} + \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Var}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \\
&= \sum_{h=1}^{\infty} h^{-\left(1+\tilde{\delta}\right)} \mathcal{O}(1) + \mathcal{O}(1) \\
&= \mathcal{O}(1),
\end{aligned}
$$

and the proof is closed. □

Equipped with the covariance results, we are now able to turn our attention to the proof of the CLT. Said proof relays on a CLT for triangular arrays of m-dependent random variables established by Romano and Wolf (2000). However, the structure of the proof is based on the one used in Jentsch et al. (2020).

Proof of Theorem 2.15. Throughout the proof we use the following notation based on (2.15): Define

$$
Y_T\left(\underline{s}\right) := \sum_{t=1}^{T} w_{t,T}\, \bar{f}\left(\underline{s}, \underline{X}_{t,T}\right) \quad \text{and} \quad Y_{t,T} := \underline{c}'\left(w_{t,T}\, \bar{f}\left(\underline{s}_j, \underline{X}_{t,T}^{(M)}\right), j = 1, \dots, J\right) \tag{A.17}
$$

for $t = 1, \dots, T$, where $\left(\underline{X}_{t,T}^{(M)}\right)_{t=1}^{T}$ is, as before, the truncated version of $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ for some $M \in \mathbb{N}$ as seen in (2.12). In addition to that, we consider

$$
Z_T := \underline{c}'\left(Y_T\left(\underline{s}_j\right), j = 1, \dots, J\right) \quad \text{and} \quad Z_T^{(M)} := \sum_{t=1}^{T} Y_{t,T}.
$$

Let

$$
\mathbf{V}_M := \left(\mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right)\right)_{j_1, j_2 = 1, \dots, J}
$$

denote the corresponding truncated version of the covariance matrix, that is

$$\mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right)$$
$$:= \sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right).$$

Note that the covariance matrix $\mathbf{V}_M$ is defined completely analogue to $\mathbf{V}$ but based on $\left(\underline{X}_{t,T}^{(M)}\right)_{t=1}^{T}$ with use of the truncated version of the companion process as described in (2.13). Later on, we will benefit from the altered dependence structure within the truncated process. Returning to the initial result in question, we have to show

$$\left(Y_T\left(\underline{s}_j\right), j = 1, \dots, J\right) \xrightarrow{d} \mathcal{N}(\underline{0}, \mathbf{V})$$

as T tends to ∞ using the newly introduced notation. This, in turn, is equivalent to showing for all $\underline{c} \in \mathbb{R}^J$ the corresponding CLT

$$Z_T \xrightarrow{d} \mathcal{N}\left(0, \underline{c}'\mathbf{V}\underline{c}\right) \tag{A.18}$$

for $T \to \infty$ by the application of the Cramér-Wold theorem. Before we go more into detail, we have to distinguish between two cases concerning the variance:

$$\text{a) } \underline{c}'\mathbf{V}\underline{c} = 0 \qquad \text{and} \quad \text{b) } \underline{c}'\mathbf{V}\underline{c} > 0.$$

In the following, we address these two cases one-by-one:

a) Let $\underline{c}'\mathbf{V}\underline{c} = 0$. This is the case if $\underline{c} \in \ker(\mathbf{V})$ holds with $\ker(\mathbf{V}) := \left\{\underline{a} \in \mathbb{R}^J \,\middle|\, \underline{a}'\mathbf{V}\underline{a} = 0\right\}$. If $\mathbf{V}$ is positive definite, trivially, we have $\ker(\mathbf{V}) = \{\underline{0}\}$. However, if $\mathbf{V}$ is singular, it holds $\dim\left(\ker(\mathbf{V})\right) \geq 1$. Now consider $\underline{c} \in \ker(\mathbf{V})$. Thus, we have

$$\operatorname{Var}\left(Z_T\right) \longrightarrow \underline{c}'\mathbf{V}\underline{c} = 0$$

for T tending to ∞, and therefore it holds

$$Z_T \xrightarrow{d} \delta_0$$

as $T \to \infty$, where δ_0 denotes the Dirac measure centered on 0, which represents the almost sure outcome in 0. Because of $\delta_0 \overset{d}{=} N(0,0)$, the first case is finished.

b) Let $\underline{c}'\mathbf{V}\underline{c} > 0$. In order to show (A.18), Proposition 6.3.9 of Brockwell and Davis (1991) imposes the verification of the following conditions:

(1) $\forall M \in \mathbb{N} : Z_T^{(M)} \xrightarrow[T\to\infty]{d} \mathcal{N}\left(0, \underline{c}'\mathbf{V}_M\underline{c}\right)$,

(2) $\underline{c}'\mathbf{V}_M\underline{c} \xrightarrow[M\to\infty]{} \underline{c}'\mathbf{V}\underline{c}$,

(3) $\forall \epsilon > 0 : \lim_{M\to\infty} \limsup_{T\to\infty} P\left(\left|Z_T - Z_T^{(M)}\right| \geq \epsilon\right) = 0.$

We start with constraint (2). It holds $\underline{c}'\mathbf{V}\underline{c} - \underline{c}'\mathbf{V}_M\underline{c} = \underline{c}'\left(\mathbf{V} - \mathbf{V}_M\right)\underline{c}$. Hence, it suffices to look at $(\mathbf{V} - \mathbf{V}_M)$, and we have to show

$$\mathbf{V}\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) - \mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) \longrightarrow 0$$

as M tends to ∞ for $j_1, j_2 = 1, \ldots, J$. The difference above can be bounded by

$$\begin{aligned}
&\left|\mathbf{V}\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) - \mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right)\right| \\
&= \Bigg|\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\
&\quad - \sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg| \\
&\leq \left|\sum_{|h|>2M-1} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\quad + \Bigg|\sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \Bigg(\operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\
&\qquad\qquad - \operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg)\Bigg| \\
&=: \underline{\mathrm{I}} + \underline{\mathrm{II}}.
\end{aligned} \tag{A.19}$$

Now we examine the newly defined terms in (A.19) individually starting with the first. Using version (a) of Lemma 2.11, we get

$$\begin{aligned}
\underline{\mathrm{I}} &\leq \sum_{|h|>2M-1} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \left|\operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq \sum_{|h|>2M-1} \frac{C_{Cov}}{|h|^{1+\tilde{\delta}}},
\end{aligned}$$

which tends to 0 as M tends to ∞. For the second summand in (A.19), on the other hand, it holds

$$\begin{aligned}
\underline{\mathrm{II}} \leq \sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \Bigg|&\operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\
&- \operatorname{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg|.
\end{aligned} \tag{A.20}$$

The next step is to derive an upper bound for the covariance difference. To this end, we disassemble the two covariances in order to bound differences of expectations instead. In doing so, we obtain

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right.\\
&\qquad\left.-\operatorname{Cov}\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right),f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&=\left|E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right.\\
&\qquad-E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\\
&\qquad-E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\\
&\qquad\left.+E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\leq\left|E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right.\\
&\qquad\left.-E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\qquad+\left|E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right.\\
&\qquad\left.-E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&=:\underline{\mathrm{II}}\mathrm{a}+\underline{\mathrm{II}}\mathrm{b}.
\end{aligned}\tag{A.21}$$

As yet seen in the previous proofs, we want to make use of the closeness between the companion process and its truncated version. Therefore, we need to rewrite the two summands of equation (A.21). Starting with the second, that is $\underline{\mathrm{II}}\mathrm{b}$, it holds

$$\begin{aligned}
\underline{\mathrm{II}}\mathrm{b}&=\left|E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right.\\
&\qquad\left.+E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\leq\left|E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right|\\
&\qquad+\left|E\left(f\left(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)E\left(f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&=:\underline{\mathrm{II}}\mathrm{ba}+\underline{\mathrm{II}}\mathrm{bb},
\end{aligned}\tag{A.22}$$

whereof the first subterm can be bounded as follows using Assumption 2, the Lipschitz condition (2.11) and Lemma 2.9:

$$\begin{aligned}\underline{\text{II}}\text{ba} &\leq E\left|f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right| E\left|f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right| \\ &\leq C\left\|\underline{\widetilde{X}}_0\left(\frac{t}{T}\right) - \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right\|_1 \\ &\leq C\left\|\underline{\varepsilon}_0\right\|_1 d\sum_{|j|\geq M}\frac{B}{l(j)}.\end{aligned}$$

In the same manner, we obtain the very same bound for the second subterm in (A.22). Consequently, subterm $\underline{\text{II}}$b can be bounded by

$$C\left\|\underline{\varepsilon}_0\right\|_1 d\sum_{|j|\geq M}\frac{B}{l(j)}. \tag{A.23}$$

The remaining summand of (A.21), namely IIa, can be rewritten as

$$\begin{aligned}&\Bigg|E\Bigg(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) \\ &\qquad\qquad - f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\Bigg)\Bigg| \\ &= \Bigg|E\Bigg(\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right) f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) \\ &\qquad + f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\left(f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg)\Bigg| \\ &\leq \left|E\left(\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right) f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\ &\qquad + \left|E\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\left(f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right)\right| \\ &=: \underline{\text{II}}\text{aa} + \underline{\text{II}}\text{ab}.\end{aligned}$$

Next, we are able to bound $\underline{\text{II}}$aa with use of Lemma 2.10 via

$$\begin{aligned}\underline{\text{II}}\text{aa} &\leq E\left|\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right) f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right| \\ &\leq C_{DP}\left(\left\|\underline{\varepsilon}_0\right\|_1 d\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}},\end{aligned}$$

and, analogously, we have

$$\underline{\text{II}}\text{ab} \le C \left(\|\underline{\varepsilon}_0\|_1 \, d \sum_{|j| \ge M} \frac{B}{l(j)} \right)^{\frac{\delta}{1+\delta}}$$

since both the expectation of f containing the companion process and the one comprising the truncated version are bounded. Hence, $\underline{\text{II}}$a is bounded from above by

$$C \left(\|\underline{\varepsilon}_0\|_1 \, d \sum_{|j| \ge M} \frac{B}{l(j)} \right)^{\frac{\delta}{1+\delta}} . \tag{A.24}$$

Eventually, combining the bounds established in A.23 and (A.24) we can bound the covariance difference as follows:

$$\begin{aligned} &\Bigg| \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\ &\qquad\qquad - \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right) \Bigg| \\ &\le C \left(\left(\|\underline{\varepsilon}_0\|_1 \, d \sum_{|j| \ge M} \frac{B}{l(j)} \right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 \, d \sum_{|j| \ge M} \frac{B}{l(j)} \right) . \end{aligned} \tag{A.25}$$

At this point, we remember the calculations made in the proof pertaining to version (a) of Lemma 2.11 and get

$$\begin{aligned} &\Bigg| \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\ &\qquad\qquad - \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right) \Bigg| \\ &\le \frac{C}{M^{1+\tilde{\delta}}} \end{aligned}$$

instead of (A.25). Now we go back to equation (A.20) and obtain inserting the newly established bound

$$\begin{aligned} \sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} &\Bigg| \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) \\ &\qquad - \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right) \Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T}\, \frac{C}{M^{1+\tilde{\delta}}} \\
&\leq C \sum_{h=-2(M-1)}^{2(M-1)} M^{-\left(1+\tilde{\delta}\right)} \\
&= C\, \frac{4M-3}{M^{1+\tilde{\delta}}} \\
&= \mathcal{O}\left(M^{-\tilde{\delta}}\right).
\end{aligned}
$$

As $\mathcal{O}\left(M^{-\tilde{\delta}}\right)$ is part of the class $o(1)$, this finishes the verification of condition (2). Below, we focus on condition (1). Since it holds $\underline{c}'\mathbf{V}\underline{c} > 0$ and $\underline{c}'\mathbf{V}_M\underline{c}$ converges to $\underline{c}'\mathbf{V}\underline{c}$ as M tends to ∞, it holds $\underline{c}'\mathbf{V}_M\underline{c} > 0$ for sufficiently large M. Hence, it is adequate to show

$$
\frac{Z_T^{(M)}}{\left(\operatorname{Var}\left(Z_T^{(M)}\right)\right)^{1/2}} \xrightarrow{d} \mathcal{N}(0,1) \tag{A.26}
$$

for $T \to \infty$ because

$$
\operatorname{Var}\left(Z_T^{(M)}\right) = \underline{c}'\mathbf{V}_M\underline{c} + o(1) \tag{A.27}
$$

can be demonstrated analogously to Lemma 2.13. As the number of non-zero weights equals d_T^{-1}, $Z_T^{(M)}$ has only d_T^{-1} non-vanishing summands denoted by $Y_{t_1,T}, \ldots, Y_{t_{d_T^{-1}},T}$. Consequently, we have

$$
Z_T^{(M)} = \sum_{h=1}^{d_T^{-1}} Y_{t_h,T}, \tag{A.28}
$$

and $\left(Y_{t_h,T}, h = 1, \ldots, d_T^{-1}\right)$ form a triangular array of centered $(2(M-1))$-dependent random variables such that the CLT in Theorem 2.1 in Romano and Wolf (2000) can be applied if the requirements listed therein can be fulfilled. These conditions read for $\delta > 0$ and $-1 \leq \gamma < 1$ as well as finite constants $\Delta_T, K_{1,T}$ and $K_{2,T}$ depending on T as follows:

(i) $E\left|Y_{t_h,T}\right|^{2+\delta} \leq \Delta_T \quad \forall\, h \in \left\{1, \ldots, d_T^{-1}\right\}$,

(ii) $\frac{\operatorname{Var}\left(\sum_{h=a}^{a+k-1} Y_{t_h,T}\right)}{k^{1+\gamma}} \leq K_{1,T} \quad \forall\, a\, \forall\, k \geq 2(M-1)$,

(iii) $\frac{\operatorname{Var}\left(\sum_{h=1}^{d_T^{-1}} Y_{t_h,T}\right)}{d_T^{-1}(2(M-1))^{\gamma}} \geq K_{2,T}$,

(iv) $\frac{K_{1,T}}{K_{2,T}} = \mathcal{O}(1)$,

(v) $\frac{\Delta_T}{K_{2,T}^{1+\delta/2}} = \mathcal{O}(1)$,

(vi) $\frac{(2(M-1))^{1+(1-\gamma)(1+2/\delta)}}{d_T^{-1}} \underset{T\to\infty}{\longrightarrow} 0.$

As these requirements only have to be valid for some γ situated in $[-1,1)$, we assume γ to be equal to zero. Then, the conditions (ii), (iii) and (vi) reduce themselves to

(ii′) $\frac{\mathrm{Var}\left(\sum_{h=a}^{a+k-1} Y_{t_h,T}\right)}{k} \leq K_{1,T} \quad \forall\, a \,\forall\, k \geq 2(M-1),$

(iii′) $\frac{\mathrm{Var}\left(\sum_{h=1}^{d_T^{-1}} Y_{t_h,T}\right)}{d_T^{-1}} \geq K_{2,T},$

(vi′) $\frac{(2(M-1))^{2+2/\delta}}{d_T^{-1}} \underset{T\to\infty}{\longrightarrow} 0.$

In the following lines, we verify the validity of the conditions stated above starting with (i). With regard to the required moments, it holds

$$\begin{aligned}
E\left|Y_{t_h,T}\right|^{2+\delta} &= E\left|\underline{c}'\left(w_{t_h,T}\,\bar{f}\left(\underline{s}_j, \underline{X}_{t_h,T}^{(M)}\right), j=1,\ldots,J\right)\right|^{2+\delta} \\
&\leq w_{t_h,T}^{2+\delta}\, E\left(\sum_{j=1}^{J} |c_j| \left|\bar{f}\left(\underline{s}_j, \underline{X}_{t_h,T}^{(M)}\right)\right|\right)^{2+\delta} \\
&= w_{t_h,T}^{2+\delta} \left\|\sum_{j=1}^{J} |c_j| \left|\bar{f}\left(\underline{s}_j, \underline{X}_{t_h,T}^{(M)}\right)\right|\right\|_{2+\delta}^{2+\delta} \\
&\leq w_{t_h,T}^{2+\delta} \left(\sum_{j=1}^{J} |c_j| \left\|\bar{f}\left(\underline{s}_j, \underline{X}_{t_h,T}^{(M)}\right)\right\|_{2+\delta}\right)^{2+\delta} \\
&\leq C\, d_T^{\frac{2+\delta}{2}} \left(\sum_{j=1}^{J} |c_j|\right)^{2+\delta}
\end{aligned}$$

as f has finite $(2+\delta)$-th absolute moments. Defining

$$\Delta_T := C\, d_T^{\frac{2+\delta}{2}} \left(\sum_{j=1}^{J} |c_j|\right)^{2+\delta}$$

provides the desired bound since Δ_T does not depend on h. Thus, condition (i) using the definition of Δ_T as stated above holds true for all $h \in \left\{1,\ldots,d_T^{-1}\right\}$. Going on to requirement (ii′), we transform the variance of the sum into sums of covariances, that is

$$\mathrm{Var}\left(\sum_{h=a}^{a+k-1} Y_{t_h,T}\right) = \sum_{h_1,h_2=a}^{k} \mathrm{Cov}\left(Y_{t_{h_1+a-1},T}, Y_{t_{h_2+a-1},T}\right). \tag{A.29}$$

Now we rewrite the double sum in (A.29) to work with

$$\sum_{h=\max\{-(k-1),-2M\}}^{\min\{k-1,2M\}} \sum_{s=\max\{1,1-h\}}^{\min\{k,k-h\}} \operatorname{Cov}\left(Y_{t_{s+h+a-1},T}, Y_{t_{s+a-1},T}\right) \tag{A.30}$$

in lieu of the right-hand side of (A.29). Remembering the definition of $(Y_{t,T})_{t=1}^{T}$ (A.17), we see that pulling the weights out of the covariance in (A.30) does not change its finiteness. Therefore, making use of the upper bounds for the weights as presupposed in Assumption 3, we can bound the covariance in equation (A.30) via

$$\left|\operatorname{Cov}\left(Y_{t_{s+h+a-1},T}, Y_{t_{s+a-1},T}\right)\right| \leq C\, d_T.$$

Taking the whole expression in (A.30) back into account, this leads to

$$\operatorname{Var}\left(\sum_{h=a}^{a+k-1} Y_{t_h,T}\right) \leq (2M+1)\, C\, k\, d_T.$$

Thus, dividing both sides of the inequality by k gives

$$\frac{\operatorname{Var}\left(\sum_{h=a}^{a+k-1} Y_{t_h,T}\right)}{k} \leq (2M+1)\, C\, d_T,$$

whereof we define the right-hand side as $K_{1,T}$. Then, the newly defined $K_{1,T}$ is independent of a, and hence this bound is valid for all a. Consequently, condition (ii′) is fulfilled, and we can turn to (iii′). Regarding equations (A.27) and (A.28) leads directly to

$$\operatorname{Var}\left(\sum_{h=1}^{d_T^{-1}} Y_{t_h,T}\right) = \underline{c}'\mathbf{V}_M\underline{c} + o(1).$$

Additionally, remember for satisfactory large M it holds $\underline{c}'\mathbf{V}_M\underline{c} > 0$. Hence, for T large enough we have

$$d_T \operatorname{Var}\left(\sum_{h=1}^{d_T^{-1}} Y_{t_h,T}\right) \geq C_L\, d_T$$

for some positive constant $C_L < \infty$. Consequently, setting $K_{2,T} := C_L\, d_T$, we obtain

$$d_T \operatorname{Var}\left(\sum_{h=1}^{d_T^{-1}} Y_{t_h,T}\right) \geq K_{2,T},$$

which shows the validity of (iii′). With both $K_{1,T}$ and $K_{2,T}$ defined as above to suit (ii′) and (iii′), respectively, we are now able to verify condition (iv) since it holds

$$\frac{K_{1,T}}{K_{2,T}} = \frac{(2M+1)\,C\,d_T}{C_L\,d_T} = \frac{(2M+1)\,C}{C_L} = \mathcal{O}(1).$$

Next, we use again $K_{2,T}$ and Δ_T originating from requirement (i) and obtain

$$\frac{\Delta_T}{K_{2,T}^{1+\frac{\delta}{2}}} = \frac{C\,d_T^{\frac{2+\delta}{2}}\left(\sum_{j=1}^{J}|c_j|\right)^{2+\delta}}{(C_L\,d_T)^{1+\delta/2}} = \frac{C\left(\sum_{j=1}^{J}|c_j|\right)^{2+\delta}}{C_L^{1+\delta/2}} = \mathcal{O}(1),$$

which shows the fulfillment of condition (v). Lastly, we see that requirement (vi′) holds trivially due to the fact that M is fixed. Recapitulatory, all required conditions are given in our setup. This allows for the application of the CLT in Theorem 2.1 in Romano and Wolf (2000), which shows (A.26). Therefore, condition (1) is finished and we move on to the remaining constraint (3). Owing to Markov's inequality, we have

$$P\left(\left|Z_T - Z_T^{(M)}\right| \geq \epsilon\right) \leq \frac{E\left|Z_T - Z_T^{(M)}\right|^2}{\epsilon^2}. \tag{A.31}$$

Thus, it is enough to examine $E\left|Z_T - Z_T^{(M)}\right|^2$. To simplify the notation, we will only focus on the case $J = 1$. However, the following procedure can easily be transferred to higher values of J. We start by transforming the expectation into a variance. Since the calculations aiming to bound said variance rely on the more universal ones for the covariances, a generalization can be made using the same arguments. It holds

$$E\left|Z_T - Z_T^{(M)}\right|^2 = \operatorname{Var}\left(Z_T - Z_T^{(M)}\right) + \left(E\left(Z_T - Z_T^{(M)}\right)\right)^2 = \operatorname{Var}\left(Z_T - Z_T^{(M)}\right)$$

because both Z_T and $Z_T^{(M)}$ are centered. Thus, we consider $\operatorname{Var}\left(Z_T - Z_T^{(M)}\right)$ from now on. Since we deal with a variance term, we can eliminate the expectations included in $\bar{f}$ as these are non-stochastic. To this end, we obtain

$$\operatorname{Var}\left(Z_T - Z_T^{(M)}\right) = \operatorname{Var}\left(\sum_{t=1}^{T} c\,w_{t,T}\left(f\left(\underline{s}, \underline{X}_{t,T}\right) - f\left(\underline{s}, \underline{X}_{t,T}^{(M)}\right)\right)\right).$$

In the following, we want to use a comparable result to Lemma 2.13 but for covariances of differences such as $f\left(\underline{s}, \underline{X}_{t,T}\right) - f\left(\underline{s}, \underline{X}_{t,T}^{(M)}\right)$. The proof of said result can be executed in a similar manner as the proof belonging to Lemma 2.13 if we can bound the mentioned covariances properly. Later on, we will need to bound covariances containing another type of differences, to wit $f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)$. Since showing these bounds happens to use the same steps, we limit ourselves to the second alternative using the proof of version (a) of Lemma 2.11 as a guideline. We start by inserting another truncated version of the companion process and obtain

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right),\right.\right.\\
&\qquad\qquad\left.\left. f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\leq\left|\operatorname{Cov}\left(\left(f\left(\underline{s},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right.\right.\\
&\qquad\left.-\left(f\left(\underline{s},\underline{\widetilde{X}}_0^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M,v)}\left(\frac{t}{T}\right)\right)\right),\right.\\
&\qquad\left.\left. f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\qquad+\left|\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_0^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M,v)}\left(\frac{t}{T}\right)\right),\right.\right.\\
&\qquad\left(f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\\
&\qquad\left.\left.-\left(f\left(\underline{s},\underline{\widetilde{X}}_h^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M,v)}\left(\frac{t}{T}\right)\right)\right)\right)\right|\\
&=: \mathrm{I}+\mathrm{II}
\end{aligned}\tag{A.32}$$

with

$$\underline{\widetilde{X}}_h^{(M,v)}\left(\frac{t}{T}\right):=\underline{\mu}\left(\frac{t}{T}\right)+\sum_{|j|<\min\{M,v\}}A\left(\frac{t}{T},j\right)\underline{\varepsilon}_{t-j}$$

and $v:=\lceil|h|/2\rceil$. As the two newly defined terms I and II have a similar building type, we stick to the examination of the former and, afterwards, transfer the results to the latter. By splitting I, we have

$$\begin{aligned}
\mathrm{I}\leq&\left|\operatorname{Cov}\left(\left(f\left(\underline{s},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right.\right.\\
&\left.\left.-\left(f\left(\underline{s},\underline{\widetilde{X}}_0^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M,v)}\left(\frac{t}{T}\right)\right)\right),f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right|\\
&+\left|\operatorname{Cov}\left(\left(f\left(\underline{s},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right.\right.\\
&\left.\left.-\left(f\left(\underline{s},\underline{\widetilde{X}}_0^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M,v)}\left(\frac{t}{T}\right)\right)\right),f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
=:&\ \mathrm{Ia}+\mathrm{Ib}.
\end{aligned}\tag{A.33}$$

Now the first summand, that is Ia, can be bounded by

$$\left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}_0}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}_0^{(v)}}\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}_h}\left(\frac{t}{T}\right)\right)\right)\right|$$
$$+\left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}_0^{(M)}}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}_0^{(M,v)}}\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}_h}\left(\frac{t}{T}\right)\right)\right)\right|$$
$$=: \mathrm{Iaa} + \mathrm{Iab}. \tag{A.34}$$

Again, we continue with the first summand of (A.34) and get along the lines of the proof appurtenant to the first version of Lemma 2.11

$$\mathrm{Iaa} \leq C\left(\left(\|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq v} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq v} \frac{B}{l(j)}\right).$$

Analogously, it holds

$$\begin{aligned}\mathrm{Iab} &\leq C\left(\left(\|\underline{\varepsilon}_0\|_1 d \sum_{M>|j|\geq \min\{M,v\}} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 d \sum_{M>|j|\geq \min\{M,v\}} \frac{B}{l(j)}\right) \\ &\leq C\left(\left(\|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq v} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq v} \frac{B}{l(j)}\right).\end{aligned}$$

In consequence, combining both bounds leads to

$$\mathrm{Ia} \leq C\left(\left(\|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq v} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq v} \frac{B}{l(j)}\right).$$

Similarly, we get the same bound for Ib of equation A.33. As a result, we have

$$\mathrm{I} \leq C\left(\left(\|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq \lceil \frac{|h|}{2} \rceil} \frac{B}{l(j)}\right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 d \sum_{|j|\geq \lceil \frac{|h|}{2} \rceil} \frac{B}{l(j)}\right),$$

and using the same steps yields the very same result for II of (A.32) as well. In conclusion, we can bound the covariance difference in question in the following way

$$\left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}_0}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}_0^{(M)}}\left(\frac{t}{T}\right)\right),\right.\right.$$
$$\left.\left. f\left(\underline{s}, \underline{\widetilde{X}_h}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}_h^{(M)}}\left(\frac{t}{T}\right)\right)\right)\right|$$

$$\leq C \left(\left(\|\underline{\varepsilon}_0\|_1 \, d \sum_{|j| \geq \lceil \frac{|h|}{2} \rceil} \frac{B}{l(j)} \right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 \, d \sum_{|j| \geq \lceil \frac{|h|}{2} \rceil} \frac{B}{l(j)} \right).$$

This bound is independent of M and can be totaled over h. Following the previously made steps, we obtain the same bound for covariances dealing with differences such as $f\left(\underline{s}, \underline{X}_{t,T}\right) - f\left(\underline{s}, \underline{X}_{t,T}^{(M)}\right)$. So Lebesgue's theorem can be used to justify the following result based on the same argumentation as in the proof of Lemma 2.13:

$$\begin{aligned}
&\lim_{M\to\infty} \lim_{T\to\infty} \operatorname{Var}\left(\sum_{t=1}^{T} c\, w_{t,T} \left(f\left(\underline{s}, \underline{X}_{t,T}\right) - f\left(\underline{s}, \underline{X}_{t,T}^{(M)}\right)\right)\right) \\
&= \lim_{M\to\infty} c^2 \sum_{h=-\infty}^{\infty} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), \right. \\
&\qquad\qquad \left. f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right) \\
&= c^2 \sum_{h=-\infty}^{\infty} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \lim_{M\to\infty} \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), \right. \\
&\qquad\qquad \left. f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right).
\end{aligned} \tag{A.35}$$

With the calculations made before, we can bound (A.35) by

$$C \sum_{h=-\infty}^{\infty} \lim_{M\to\infty} \left(\left(\|\underline{\varepsilon}_0\|_1 \, d \sum_{|j|\geq M} \frac{B}{l(j)} \right)^{\frac{\delta}{1+\delta}} + \|\underline{\varepsilon}_0\|_1 \, d \sum_{|j|\geq M} \frac{B}{l(j)} \right) = 0.$$

Now we go back to (A.31) and obtain

$$\lim_{M\to\infty} \limsup_{T\to\infty} P\left(\left| Z_T - Z_T^{(M)} \right| \geq \epsilon \right) \leq \lim_{M\to\infty} \limsup_{T\to\infty} \frac{E\left(\left| Z_T - Z_T^{(M)} \right|^2 \right)}{\epsilon^2} = 0$$

for all $\epsilon > 0$, which concludes (c) and hence finishes the proof.

$\square$

A.1.3. Proofs of Section 2.3

This subsection finalises the proofs belonging to Chapter 2 by showing the tightness result followed by the FCLT. Nevertheless, we start with the preparatory result for the tightness, namely Lemma 2.17. Its proof is displayed in the following lines:

Proof of Lemma 2.17. Since both cases (a) and (b) can be treated in the same way, we do not distinguish between them during the proof. We start by adding self-canceling differences in order to bring the companion process in, to wit

$$
\begin{aligned}
&P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)\right| > \lambda\right)\\
&\leq P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)-\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right)\right| > \frac{\lambda}{2}\right)\\
&\qquad\qquad + P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| > \frac{\lambda}{2}\right).
\end{aligned}
$$

Now we aim to show that the second summand can be neglected if T tends to infinity. Using Markov's inequality and Lemma 2.8, we can bound the said summand via

$$
\begin{aligned}
&P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)-\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right)\right| > \frac{\lambda}{2}\right)\\
&\leq \frac{2}{\lambda}\,E\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)-\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right)\right|\right)\\
&\leq C\,d_T^{1/2},
\end{aligned}
$$

which tends to 0 if $T\to\infty$. Hence, we have

$$
\begin{aligned}
&P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)\right| > \lambda\right)\\
&\qquad\qquad \leq P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| > \frac{\lambda}{2}\right) + C\,d_T^{1/2}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\lim_{T\to\infty} P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)\right| > \lambda\right)\\
&\qquad\qquad = \lim_{T\to\infty} P\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| > \frac{\lambda}{2}\right),
\end{aligned}
$$

which is the desired result. $\square$

The above-written proof allows us to proceed with the verification of the stated tightness result. At this, the procedure is inspired by Arcones and Yu (1994) in combination with the extensions made in Jentsch et al. (2020).

Proof of Lemma 2.18. Since the first part of the proof is similar in both cases (a) and (b), we split the argumentation later and begin with the introduction of the general framework. For this purpose, we define

$$\kappa_T := \left\lfloor d_T^{-\frac{1}{2m}} \right\rfloor \qquad \text{and} \qquad \mu_T := \left\lfloor \frac{d_T^{-1}}{2\kappa_T} \right\rfloor . \tag{A.36}$$

We know that d_T^{-1} denotes the number of positive weights, but the non-vanishing weights need not to be subsequent. In the style of Arcones and Yu (1994) we divide our set of indices into blocks H_t, T_t and R, which do not need to be equally long. In fact, we choose the blocks in such a way that the indices of the first κ_T non-negative weights are in H_1, the indices of the second κ_T non-negative weights in T_1, the indices of the second κ_T non-negative weights in H_2 and so on until we have eventually μ_T H-blocks and μ_T T-blocks each. The remaining indices are arranged in block R. Note that the number of indices corresponding to non-negative weights and situated in R is smaller than κ_T.

Having the definition of the truncated process, equation (2.13), in mind, we can establish an upper bound for (2.19) the following way:

$$\begin{aligned}
&\lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{T} w_{t,T} \left(\bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_t\left(\frac{t}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_t\left(\frac{t}{T}\right)\right)\right) \right| > \lambda \right) \\
&\leq \lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \right.\right. \\
&\qquad\qquad \left.\left. \cdot \left(\bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right) \right| > \frac{\lambda}{3} \right) \\
&\qquad\qquad + \lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{\mu_T} \sum_{i\in T_t} w_{i,T} \right.\right. \\
&\qquad\qquad \left.\left. \cdot \left(\bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right) \right| > \frac{\lambda}{3} \right) \\
&\quad + \lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{i\in R} w_{i,T} \left(\bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right) \right| > \frac{\lambda}{3} \right) \\
&=: \underline{\mathrm{I}} + \underline{\mathrm{II}} + \underline{\mathrm{III}}.
\end{aligned} \tag{A.37}$$

The last term from above can be bounded using the Lipschitz condition (2.11) and Assumption 3, and we obtain

$$
\begin{aligned}
\underline{\underline{\underline{\mathrm{I}}}} &\leq \lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \sum_{i\in R} w_{i,T} \left| \bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right| > \frac{\lambda}{3}\right) \\
&\leq \lim_{r\to 0} \limsup_{T\to\infty} P\left(\frac{3}{\lambda} \sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \sum_{i\in R} w_{i,T} \left| \bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right| > 1\right) \\
&\leq \lim_{r\to 0} \limsup_{T\to\infty} E\left(C \sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \sum_{i\in R} w_{i,T} \left| \bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right| \right) \\
&\leq \lim_{r\to 0} \limsup_{T\to\infty} C \sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \sum_{i\in R} w_{i,T}\, E\left(g\left(\widetilde{\underline{X}}_i\left(\frac{i}{T}\right), \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right) \rho\left(\underline{s}_1, \underline{s}_2\right) \\
&\leq \limsup_{T\to\infty} C \sum_{i\in R} w_{i,T} \\
&\leq \limsup_{T\to\infty} C\, d_T^{1/2} \kappa_T \\
&\leq \limsup_{T\to\infty} C\, d_T^{1/2} d_T^{-\frac{1}{2m}} \\
&= \limsup_{T\to\infty} C\, d_T^{\frac{m-1}{2m}} \\
&= 0
\end{aligned}
$$

since it holds $m > 1$ and $\operatorname{card}(R) < \kappa_T$.

As $\underline{\underline{\mathrm{I}}}$ of (A.37) can be treated analogously to $\underline{\mathrm{I}}$, we focus on $\underline{\mathrm{I}}$. In the following, we want to make use of the block structure in such a way that the involved random variables whose indices are situated in different blocks $H_1, \ldots, H_{\mu_T}$ are independent. To achieve this, we make use of the truncated version $\left(\widetilde{\underline{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)$ of $\left(\widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)$ defined in (2.13) with $M = \lceil \kappa_T/2 \rceil$. First, we insert the truncated process in $\underline{\mathrm{I}}$ with the help of self-canceling terms in order to generate subterms, which we examine individually afterwards. We get

$$
\begin{aligned}
\underline{\mathrm{I}} = \lim_{r\to 0} \limsup_{T\to\infty} P\Bigg(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \Bigg| \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} & \left(\bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i\left(\frac{i}{T}\right)\right)\right) \\
& - \left(\bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \\
& + \left(\bar{f}\left(\underline{s}_1, \widetilde{\underline{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \widetilde{\underline{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \Bigg| > \frac{\lambda}{3}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \right.\right.\\
&\qquad \left.\left. \cdot \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right| > \frac{\lambda}{9}\right)\\
&\qquad + \lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \right.\right.\\
&\qquad \left.\left. \cdot \left(\bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right| > \frac{\lambda}{9}\right)\\
&\qquad + \lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \right.\right.\\
&\qquad\qquad \left.\left. \cdot \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right| > \frac{\lambda}{9}\right)\\
&=: \underline{\mathrm{I}}\mathrm{a} + \underline{\mathrm{I}}\mathrm{b} + \underline{\mathrm{I}}\mathrm{c}.
\end{aligned}
\tag{A.38}
$$

Using Markov's inequality and then Lemma 2.3, it holds for $\underline{\mathrm{I}}\mathrm{a}$

$$
\begin{aligned}
\underline{\mathrm{I}}\mathrm{a} &= \lim_{r\to 0} \limsup_{T\to\infty} P\left(\frac{9}{\lambda} \sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \right.\\
&\qquad \left. \cdot \left| \bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) \right| > 1\right)\\
&\leq \lim_{r\to 0} \limsup_{T\to\infty} E\left(C \sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \left| \bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) \right| \right)\\
&\leq \limsup_{T\to\infty} C \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T}\, E \left| \underline{\widetilde{X}}_i\left(\frac{i}{T}\right) - \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right) \right|\\
&\leq \limsup_{T\to\infty} C \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \sum_{|j|\geq M} \frac{1}{l(k)}\\
&\leq \limsup_{T\to\infty} C \sum_{t=1}^{T} w_{t,T}\, \kappa_T^{-1} \sum_{|j|\geq\lceil \kappa_T/2\rceil} \frac{j^m}{l(j)}\\
&= 0.
\end{aligned}
$$

The subterm $\underline{\mathrm{I}}\mathrm{b}$ can be bounded similarly, and henceforward we focus on the last subterm of (A.38), that is

$$\lim_{r\to 0} \limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left| \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T} \right.\right.$$
$$\left.\left. \cdot \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right| > \frac{\lambda}{9}\right). \quad \text{(A.39)}$$

Since we only deal with the truncated version of process now, we obtained independence of the summands with different indices. This opens the way to the use of standard empirical process theory. Before pursuing the proof, we introduce some further notation on the basis of Arcones and Yu (1994). Consider $\underline{s}, \underline{s}_1, \underline{s}_2 \in [-S,S]^d$ and define both $\nu_T(\underline{s})$ and $\nu_T(\underline{s}_1, \underline{s}_2)$ by

$$\nu_T(\underline{s}) := \sum_{t=1}^{\mu_T} \sum_{i\in H_t} w_{i,T}\, \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) \quad \text{and} \quad \nu_T(\underline{s}_1, \underline{s}_2) := \nu_T(\underline{s}_1) - \nu_T(\underline{s}_2),$$

respectively. Influenced by Arcones and Yu (1994), we use a classical chaining argument. For this purpose, let

$$r_k := r\, 2^{-k} \quad \text{(A.40)}$$

for $k = 0, \ldots, k_T$ be a decreasing sequence for some r which will be specified thereinafter as well as the existence and order of r_{k_T}. Moreover, let $\mathcal{F}_k \subseteq [-S,S]^d$ for $k \in \{0, \ldots, k_T\}$ be an index set satisfying

$$\#\mathcal{F}_k = D(k) = D\left(r_k, [-S,S]^d, \rho\right) \qquad \text{and} \qquad \sup_{\underline{s}_1\in[-S,S]^d} \min_{\underline{s}_2\in\mathcal{F}_k} \rho(\underline{s}_1, \underline{s}_2) < r_k$$

with $D\left(u, [-S,S]^d, \rho\right) = \max\left\{\#\mathcal{T}_0 \,\middle|\, \mathcal{T}_0 \subseteq [-S,S]^d,\ \rho(\underline{s}_1, \underline{s}_2) > u \ \forall\, \underline{s}_1 \neq \underline{s}_2 \in \mathcal{T}_0\right\}$ denoting the usual packing number defined e.g. in Definition 2.2.3 of van der Vaart and Wellner (2000). We have

$$D\left(u, [-S,S]^d, \rho\right) \leq \left(\frac{2\,Sd}{u} + 1\right)^d \quad \text{(A.41)}$$

for $u > 0$. This can be seen in the following. Consider $\mathcal{T}_0 \subseteq [-S,S]^d$ satisfying $\rho(\underline{s}_1, \underline{s}_2) > u \ \forall\, \underline{s}_1 \neq \underline{s}_2 \in \mathcal{T}_0$. Hence, all sets $\underline{s} + \left(\frac{u}{2\,Sd}\right)[-S,S]^d$ with $\underline{s} \in \mathcal{T}_0$ are pairwise disjoint and contained in $\left(1 + \frac{u}{2\,Sd}\right)[-S,S]^d$. Thus, it holds

$$\sum_{\underline{s}\in\mathcal{T}_0} \lambda\!\!\lambda\left(\underline{s} + \left(\frac{u}{2\,Sd}\right)[-S,S]^d\right) \leq \lambda\!\!\lambda\left(\left(1 + \frac{u}{2\,Sd}\right)[-S,S]^d\right)$$

with $\lambda\!\!\lambda(\cdot)$ denoting the Lebesgue-measure. This means

$$\operatorname{card}(\mathcal{T}_0)\left(\frac{u}{2\,Sd}\right)^d \lambda\!\!\lambda\left([-S,S]^d\right) \leq \left(1 + \frac{u}{2\,Sd}\right)^d \lambda\!\!\lambda\left([-S,S]^d\right).$$

Hence, we obtain

$$\operatorname{card}(\mathcal{T}_0) \leq \left(1 + \frac{u}{2\,Sd}\right)^d \left(\frac{2\,Sd}{u}\right)^d = \left(\frac{2\,Sd}{u} + 1\right)^d,$$

which proves (A.41). Therefore, it holds $D(k) = \mathcal{O}\left(r_k^{-d}\right)$. This gives us the existence of maps $\pi_k \colon [-S,S]^d \to \mathcal{F}_k$ for $k = 0, \ldots, k_T$ such that

$$\left|\underline{s} - \pi_k\, \underline{s}\right|_1 \leq r_k \quad \forall \underline{s} \in [-S,S]^d.$$

Subsequently, we get the following two inequalities for $\underline{s}_1, \underline{s}_2 \in [-S,S]^d$ with $\rho\left(\underline{s}_1, \underline{s}_2\right) < r$:

$$\rho\left(\pi_0\, \underline{s}_1, \pi_0\, \underline{s}_2\right) \leq \rho\left(\pi_0\, \underline{s}_1, \underline{s}_1\right) + \rho\left(\underline{s}_1, \underline{s}_2\right) + \rho\left(\underline{s}_2, \pi_0\, \underline{s}_2\right) = 3\, r$$

and for $k \in \{1, \ldots, k_T\}$ and $\underline{s} \in [-S,S]^d$

$$\rho\left(\pi_k\, \underline{s}, \pi_{k-1}\, \underline{s}\right) \leq \rho\left(\pi_k\, \underline{s}, \underline{s}\right) + \rho\left(\underline{s}, \pi_{k-1}\, \underline{s}\right) \leq r_k + r\, 2^{-k+1} = 3\, r_k.$$

Using the setup from above, we get

$$\begin{aligned}
&\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} \left|\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right| \\
&\leq \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} \left(\left|\nu_T\left(\underline{s}_1\right) - \nu_T\left(\underline{s}_2\right) - \nu_T\left(\pi_{k_T}\, \underline{s}_1\right) + \nu_T\left(\pi_{k_T}\, \underline{s}_2\right)\right| + \left|\nu_T\left(\pi_0 \underline{s}_1\right) - \nu_T\left(\pi_0 \underline{s}_2\right)\right|\right) \\
&\qquad + \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} \left|\sum_{k=1}^{k_T} \nu_T\left(\pi_k\, \underline{s}_1\right) - \nu_T\left(\pi_{k-1}\, \underline{s}_1\right) - \nu_T\left(\pi_k\, \underline{s}_2\right) + \nu_T\left(\pi_{k-1}\, \underline{s}_2\right)\right| \\
&\leq 2 \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} \left|\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right| + \sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} \left|\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right| + 2 \sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} \left|\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right|.
\end{aligned}$$

Before we split our proof into two, we introduce some auxiliary quantities, which are needed in both cases (a) and (b). To this end, let

$$\lambda_k := r_k^{\frac{1}{4(1+\delta)}} \vee \left(\frac{4}{\bar{C}}\, r_k^{\frac{1}{1+\delta}} \log D(k)\right)^{1/2} \tag{A.42}$$

be defined for $k = 1, \ldots, k_T$ and some finite constant $\bar{C} > 0$, which will be specified later on and may take different values in cases (a) and (b). Hence, we get

$$\lambda_k \geq \left(\frac{4}{\bar{C}}\, r_k^{\frac{1}{1+\delta}} \log D(k)\right)^{1/2},$$

which is equivalent to

$$\log D(k) \leq \lambda_k^2\, \frac{\bar{C}}{4}\, r_k^{-\frac{1}{1+\delta}}. \tag{A.43}$$

Additionally, let r be small enough to allow for

$$2 \sum_{k \in \mathbb{N}} \lambda_k \leq \frac{\lambda}{27}. \tag{A.44}$$

Since we have $D(k) = \mathcal{O}\left(r_k^{-d}\right)$, the summability of $(\lambda_k)_{k=1}^{k_T}$ for $T \to \infty$ is assured. At this point, we come back to equation (A.39). With the preassigned notation and equation (A.44), we can rewrite (A.39) as follows:

$$\begin{aligned}
&P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left|\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right| > \frac{\lambda}{9}\right)\\
&= P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{9}\right)\\
&\leq P\left(2\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{27}\right)\\
&\quad + P\left(2\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > 2\sum_{k=1}^{k_T}\lambda_k\right) + P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{27}\right)\\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad\qquad (A.45)
\end{aligned}$$

In the following, we treat the individual terms in two different ways. In order to show the asymptotic negligibility of terms II and III, we want to make use of Bernstein's inequality for sums of independent random variables exerted on the outer sum of ν_T. Subsequently, term I will be discussed by using a symmetrization lemma.

As already insinuated, we arrived at the point where the different assumptions play a role. Therefore, we continue with case (a) before turning our attention to case (b).

(a) Before starting with the examination of term II in (A.45), we precise the definition of the sequence in (A.40). We need r_{k_T} to meet the following bounding condition:

$$d_T^{\frac{(2m\delta-4-3\delta)(1+\delta)}{2m(4+3\delta)}} \leq r_{k_T} \leq d_T^{\frac{2}{m(1-\delta)}}. \qquad (A.46)$$

The existence of such a k_T is guaranteed if the lower bound is smaller than the upper one. Equality is not enough since we want k_T to be an integer. The fact that our choice of m assures the presence of a real gap between the bounds can be seen in the following. The necessary inequality

$$d_T^{\frac{(2m\delta-4-3\delta)(1+\delta)}{2m(4+3\delta)}} < d_T^{\frac{2}{m(1-\delta)}}$$

is equivalent to

$$(2m\delta - 4 - 3\delta)\cdot\left(1-\delta^2\right) > 4(4+3\delta),$$

which can be rewritten as

$$m\left(2\delta - 2\delta^3\right) > 20 + 15\delta - 4\delta^2 - 3\delta^3.$$

The equation above leads to the following lower bound for m

$$m > \frac{20 + 15\delta - 4\delta^2 - 3\delta^3}{2\delta\left(1-\delta^2\right)},$$

which corresponds to our choice of m.

After the existence of k_T is ensured, we turn our attention to the second summand in (A.45). To be able to apply Bernstein's inequality, we need to establish an upper bound for the variance of the inner sum of ν_T. Consider $l := |i_1 - i_2|$. Then, we have

$$\begin{aligned}
&\operatorname{Var}\left(\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right) \\
&= \operatorname{Var}\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right) \\
&= \sum_{t=1}^{\mu_T}\operatorname{Var}\left(\sum_{i\in H_t} w_{i,T}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right) \\
&= \sum_{t=1}^{\mu_T}\sum_{i_1,i_2\in H_t} w_{i_1,T}\, w_{i_2,T}\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right),\right. \\
&\qquad\qquad \left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right) \\
&\le \sum_{t=1}^{\mu_T}\sum_{i_1,i_2\in H_t} w_{i_1,T}\, w_{i_2,T}\left|\operatorname{Cov}\left(\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)\right.\right. \\
&\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right), \\
&\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\quad + \sum_{t=1}^{\mu_T}\sum_{i_1,i_2\in H_t} w_{i_1,T}\, w_{i_2,T}\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right. \\
&\qquad \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right) \\
&\qquad \left.\left. - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M(l))}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M(l))}\left(\frac{i_2}{T}\right)\right)\right)\right)\right|
\end{aligned} \tag{A.47}$$

for $M(l) := \lceil \min\{M, l/2\}\rceil$ as truncation parameter. Next, we take a closer look at the first covariance of (A.47), since the second one behaves similarly. We have

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)\right.\right. \\
&\qquad\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right), \\
&\qquad\qquad\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq \left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right. \\
&\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\qquad + \left|\operatorname{Cov}\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|,
\end{aligned} \tag{A.48}$$

and, again, we only examine the first covariance of (A.48) due to the same reason. We continue by transforming the covariance in terms consisting of absolute moments. We get

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\right. \\
&\qquad\qquad \left. \cdot \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\qquad\qquad + E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right| \\
&\qquad\qquad\qquad\qquad \cdot E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right|.
\end{aligned} \tag{A.49}$$

In the later following calculations to bound the variance of ν_T we will need two suitable but different bounds. Therefore, we establish now two alternative bounds for (A.49). The first will make use of the closeness between the truncated and the two times truncated version of the companion process, and the second will consists of the difference between $\underline{s}_1$ and $\underline{s}_2$. At first glance, this seems to be rather complicated, but in the end we need this combination to absorb the inner sum of ν_T.

i) For the second summand of (A.49), we get

$$
\begin{aligned}
&E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right| \\
&\qquad\qquad \cdot E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right| \\
&\leq CE\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right| \\
&\leq C\left\|\underline{\varepsilon}_0\right\|_1 d \sum_{M>|j|\geq M(l)} \frac{B}{l(j)},
\end{aligned}
$$

and for the first one, it holds

$$
\begin{aligned}
&E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\right. \\
&\qquad\qquad \left.\cdot\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right) \cdot f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right| \\
&\qquad + E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right) \cdot f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right| \\
&=: \bar{\mathrm{I}} + \bar{\mathrm{II}}.
\end{aligned}
$$

Using Hoelder's inequality, we get for $\bar{\mathrm{I}}$

$$
\begin{aligned}
\bar{\mathrm{I}} &\leq \left\|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right\|_{\frac{2+\delta}{1+\delta}} \left\|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right\|_{2+\delta} \\
&\leq C\left\|\underline{\varepsilon}_0\right\|_{\frac{2+\delta}{1+\delta}} d \sum_{M>|j|\geq M(l)} \frac{B}{l(j)}
\end{aligned}
$$

and, analogously,

$$
\bar{\mathrm{II}} \leq C\left\|\underline{\varepsilon}_0\right\|_{\frac{2+\delta}{1+\delta}} d \sum_{M>|j|\geq M(l)} \frac{B}{l(j)}.
$$

Combining both bounds we get

$$
\begin{aligned}
&E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\right. \\
&\qquad\qquad \left.\cdot\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|
\end{aligned}
$$

$$\leq C \left\| \underline{\varepsilon}_0 \right\|_{\frac{2+\delta}{1+\delta}} d \sum_{M>|j|\geq M(l)} \frac{B}{l(j)},$$

and, eventually, we get for (A.49)

$$\begin{aligned}
&\left| \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right), \right.\right. \\
&\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq C \sum_{M>|j|\geq M(l)} \frac{B}{l(j)}.
\end{aligned}$$

Note that this bound stays true for $M = M(l)$ as we get an empty sum in this case.

ii) For the second summand of (A.49), we get

$$\begin{aligned}
&E\left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right| \\
&\qquad\qquad \cdot E\left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right| \\
&\leq C\rho\left(\underline{s}_1, \underline{s}_2\right)
\end{aligned}$$

by using the fact that the function f fulfills the Lipschitz condition 2.11 stated in Assumption 2. The first summand of (A.49) can be bounded via

$$\begin{aligned}
&E\left| \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right) \right. \\
&\qquad\qquad \left. \cdot \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq E\left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\qquad + E\left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&=: \breve{\mathrm{I}} + \breve{\mathrm{II}}.
\end{aligned}$$

Using again Hoelder's inequality and the Lipschitz condition 2.11, we get

$$\breve{\mathrm{I}} \leq \left\| f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right\|_{2+\delta} \left\| f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right\|_{\frac{2+\delta}{1+\delta}}$$
$$\leq C\rho\left(\underline{s}_1, \underline{s}_2\right)$$

and $\breve{\mathrm{II}} \leq C\rho\left(\underline{s}_1, \underline{s}_2\right)$ in the same manner. Together we have

$$E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\right.$$
$$\left.\cdot\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|$$
$$\leq C\rho\left(\underline{s}_1, \underline{s}_2\right)$$

and hence for (A.49)

$$\left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right.$$
$$\left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|$$
$$\leq C\rho\left(\underline{s}_1, \underline{s}_2\right)$$

since the first summand of (A.49) can be bounded by $C\rho\left(\underline{s}_1, \underline{s}_2\right)$ as well by making use of Assumption 2. This completes the second version.

The combination of these two bounds gives us

$$\left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right.$$
$$\left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|$$
$$\leq C \min\left\{\rho\left(\underline{s}_1, \underline{s}_2\right), \sum_{M > |j| \geq M(l)} \frac{B}{l(j)}\right\}.$$

As already mentioned, we get the very same bound for the second covariance of (A.48), that is

$$\left|\mathrm{Cov}\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right.$$
$$\left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|$$

$$\leq C \min \left\{ \rho\left(\underline{s}_1, \underline{s}_2\right), \sum_{M>|j|\geq M(l)} \frac{B}{l(j)} \right\}.$$

Therefore, the covariance in (A.47) can be bounded via

$$\begin{aligned}
&\left|\mathrm{Cov}\left(\left(\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)\right.\right.\right. \\
&\qquad\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right), \\
&\qquad\qquad\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq C \min \left\{ \rho\left(\underline{s}_1, \underline{s}_2\right), \sum_{M>|j|\geq M(l)} \frac{B}{l(j)} \right\}.
\end{aligned}$$

Before we investigate $\sum_{M>|j|\geq M(l)} B/l(j)$ further, note that m is larger than $(1+\delta)/\delta$. This can be seen by verifying

$$\frac{20 + 15\delta - 4\delta^2 - 3\delta^3}{2\delta\left(1-\delta^2\right)} - \frac{1+\delta}{\delta} > 0,$$

which is equivalent to $18 + 13\delta - 2\delta^2 - \delta^3 > 0$. Clearly, every $\delta \in (0,1)$ fulfills this constraint. Now we can go on with the inspection of the aforementioned sum. It holds

$$\begin{aligned}
\sum_{M>|j|\geq M(l)} \frac{B}{l(j)} &\leq \sum_{M>|j|\geq \left\lceil \frac{|i_1-i_2|}{2} \right\rceil} \frac{|j|^{\frac{1+\delta}{\delta}} |j|^{-\frac{1+\delta}{\delta}} B}{l(j)} \\
&\leq \left\lceil \frac{|i_1-i_2|}{2} \right\rceil^{-\frac{1+\delta}{\delta}} \sum_{j\in\mathbb{Z}} \frac{|j|^{\frac{1+\delta}{\delta}} B}{l(j)} \\
&\leq C\, |i_1 - i_2|^{-\frac{1+\delta}{\delta}}
\end{aligned}$$

and hence

$$\begin{aligned}
&\left|\mathrm{Cov}\left(\left(\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)\right.\right.\right. \\
&\qquad\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right), \\
&\qquad\qquad\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|
\end{aligned}$$

$$\leq C \min \left\{ \rho\left(\underline{s}_1, \underline{s}_2\right), |i_1 - i_2|^{-\frac{1+\delta}{\delta}} \right\} \tag{A.50}$$

holds true as well. Now we can go back to (A.47), and inserting the bound established in (A.50) gives us for some $R_0 \geq 2$

$$\begin{aligned}
&\operatorname{Var}\left(\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right) \\
&= \sum_{t=1}^{\mu_T} \sum_{i_1, i_2 \in H_t} w_{i_1,T}\, w_{i_2,T}\, \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right), \right. \\
&\qquad\qquad \left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right) \\
&\leq \sum_{t=1}^{\mu_T} \sum_{i_1, i_2 \in H_t} w_{i_1,T}\, w_{i_2,T}\, C \min \left\{ \rho\left(\underline{s}_1, \underline{s}_2\right), |i_1 - i_2|^{-\frac{1+\delta}{\delta}} \right\} \\
&\leq C\, \mu_T\, \kappa_T\, d_T \sum_{t=0}^{d_T^{-1}} \min \left\{ \rho\left(\underline{s}_1, \underline{s}_2\right), t^{-\frac{1+\delta}{\delta}} \right\} \\
&\leq C\, d_T^{-1}\, d_T \left(\sum_{t=0}^{R_0 - 1} \rho\left(\underline{s}_1, \underline{s}_2\right) + \sum_{t=R_0}^{d_T^{-1}} t^{-\frac{1+\delta}{\delta}} \right) \\
&\leq C \left(R_0\, \rho\left(\underline{s}_1, \underline{s}_2\right) + \int_{R_0/2}^{\infty} t^{-\frac{1+\delta}{\delta}}\, dt \right) \\
&\leq C \left(R_0\, \rho\left(\underline{s}_1, \underline{s}_2\right) + \left[t^{-1/\delta}\right]_{R_0/2}^{\infty} \right) \\
&\leq C \left(R_0\, \rho\left(\underline{s}_1, \underline{s}_2\right) + R_0^{-1/\delta} \right).
\end{aligned}$$

We set

$$R_0 := \left\lfloor \rho\left(\underline{s}_1, \underline{s}_2\right)^{-\frac{\delta}{1+\delta}} \right\rfloor,$$

and for $\rho\left(\underline{s}_1, \underline{s}_2\right) < 1$, which is ensured for r small enough, we get

$$\operatorname{Var}\left(\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right) \leq C \left(\rho\left(\underline{s}_1, \underline{s}_2\right)^{-\frac{\delta}{1+\delta}+1} + \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{1}{1+\delta}} \right) = C \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{1}{1+\delta}}.$$

Now we return to term II. Bernstein's inequality demands for bounded absolute first moments, which we cannot provide using the inner sum of ν_T as is. Therefore, the next step will be to show that bounding the product of weight and function does not affect the calculations in a negative way. It holds

$$
\begin{aligned}
&P\left(\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \sum_{k=1}^{k_T}\lambda_k\right)\\
&= P\left(\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \sum_{k=1}^{k_T}\lambda_k,\ \sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\, f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) \le d_T^{\frac{\delta}{4+3\delta}}\right)\\
&\qquad + P\left(\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \sum_{k=1}^{k_T}\lambda_k,\right.\\
&\qquad\qquad \left.\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\, f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) > d_T^{\frac{\delta}{4+3\delta}}\right)\\
&\le P\left(\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \sum_{k=1}^{k_T}\lambda_k,\ \sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\, f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) \le d_T^{\frac{\delta}{4+3\delta}}\right)\\
&\qquad + P\left(\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\, f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) > d_T^{\frac{\delta}{4+3\delta}}\right)\\
&=: \mathrm{IIa} + \mathrm{IIb}.
\end{aligned}
$$

Next, we have to assure the asymptotic negligibility of IIb. We can split IIb by

$$
\begin{aligned}
\mathrm{IIb} &= P\left(\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\left| f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{0},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\right.\right.\\
&\qquad\qquad \left.\left. + f\left(\underline{0},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\right| > d_T^{\frac{\delta}{4+3\delta}}\right)\\
&\le P\left(\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\left| f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{0},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\right| > \frac{d_T^{\frac{\delta}{4+3\delta}}}{2}\right)\\
&\qquad + P\left(\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\left| f\left(\underline{0},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\right| > \frac{d_T^{\frac{\delta}{4+3\delta}}}{2}\right)\\
&=: \mathrm{IIba} + \mathrm{IIbb}
\end{aligned}
$$

to bound each subterm individually. We get

$$\begin{aligned}
\text{IIba} &\leq P\left(\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\, g\left(\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right), \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)|\underline{s}-\underline{0}| > \frac{d_T^{\frac{\delta}{4+3\delta}}}{2}\right)\\
&\leq 2^{2+\delta}(Sd)^{2+\delta}\sum_{t=1}^{T} w_{t,T}^{2+\delta} E\left(g\left(\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right), \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\right)^{2+\delta} d_T^{-\frac{(2+\delta)\delta}{4+3\delta}}\\
&\leq d_T^{-1}\mathcal{O}\left(d_T^{\frac{2+\delta}{2}}\right) d_T^{-\frac{(2+\delta)\delta}{4+3\delta}}\\
&= \mathcal{O}\left(d_T^{\frac{\delta^2}{2(4+3\delta)}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\text{IIbb} &\leq 2^{2+\delta}\sum_{t=1}^{T} w_{t,T}^{2+\delta} E\left|f\left(\underline{0}, \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\right|^{2+\delta} d_T^{-\frac{(2+\delta)\delta}{4+3\delta}}\\
&\leq C\, d_T^{-1}\mathcal{O}\left(d_T^{\frac{2+\delta}{2}}\right) d_T^{-\frac{(2+\delta)\delta}{4+3\delta}}\\
&= \mathcal{O}\left(d_T^{\frac{\delta^2}{2(4+3\delta)}}\right).
\end{aligned}$$

Hence, it suffices to work with IIa. Consider

$$\Omega_{sup,T} := \left\{\omega\in\Omega \;\middle|\; \sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\left|f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\right| \leq d_T^{\frac{\delta}{4+3\delta}}\right\}$$

and

$$\breve{f} := \begin{cases}\bar{f}, & \omega\notin\Omega_{sup,T},\\ 0, & \omega\in\Omega_{sup,T}.\end{cases}$$

With this notation, we can apply Bernstein's inequality and get

$$\begin{aligned}
\text{IIa} \leq P\Bigg(\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}} &\Bigg|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\\
&\cdot\left(\breve{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \breve{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\Bigg| > \sum_{k=1}^{k_T}\lambda_k\Bigg)
\end{aligned}$$

$$\leq \sum_{k=1}^{k_T} \sum_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} P\left(\sum_{t=1}^{\mu_T} \left| \sum_{i \in H_t} w_{i,T} \right.\right.$$
$$\left.\left. \cdot \left(\breve{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \breve{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right| > \lambda_k \right)$$
$$\leq 2 \sum_{k=1}^{k_T} D(k) D(k-1) \exp\left(-\frac{1}{2} \cdot \frac{\lambda_k^2}{V_{II,k} + \frac{\breve{M}\lambda_k}{3}} \right), \tag{A.51}$$

where $\breve{M}$ denotes the upper bound for the first absolute moment of the sum over i and $V_{II,k}$ the one for the accumulated variances depending on k. The next step is to specify these bounds. We start with $\breve{M}$, which has to fulfil

$$\breve{M} \geq \left| \sum_{i \in H_t} w_{i,T} \left(\breve{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \breve{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right|$$

for all $\underline{s}_1, \underline{s}_2 \in [-S, S]^d$ and $t = 1, \ldots, \mu_T$. To eliminate the influence of $\underline{s}_1$ and $\underline{s}_2$, we modify the condition from above as follows:

$$\breve{M} \geq 4 \sum_{i \in H_t} d_T^{\frac{\delta}{4+3\delta}}$$

for $t = 1, \ldots, \mu_T$. Next, to eliminate the influence of t as well, we require

$$\breve{M} \geq 4\, \kappa_T\, d_T^{\frac{\delta}{4+3\delta}}$$

instead. This makes sense mainly because the summands were independent of t. By making use of the definition of κ_T, this new constraint turns into

$$\breve{M} \geq 4\, d_T^{-\frac{1}{2m}}\, d_T^{\frac{\delta}{4+3\delta}} = 4\, d_T^{\frac{2m\delta - 4 - 3\delta}{2m(4+3\delta)}}.$$

Thus, we set

$$\breve{M} := 4\, d_T^{\frac{(2-3m)\delta - 4}{2m(4+3\delta)}}.$$

Now we attend to the upper bound for the variance. Bernstein's inequality demands

$$V_{II,k} \geq \sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} \operatorname{Var}\left(\sum_{t=1}^{\mu_T} \sum_{i \in H_t} w_{i,T} \left(\breve{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \breve{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right).$$

Since we have

$$E\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right)=0,$$

it holds

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\breve{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\breve{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right)\\
&=\operatorname{Var}\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\mathbb{1}_{\Omega_{sup,T}}\right)\\
&\leq E\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\mathbb{1}_{\Omega_{sup,T}}\right)^2\\
&\leq E\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right)^2\\
&=\operatorname{Var}\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right)\\
&=\operatorname{Var}\left(\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right).
\end{aligned}\tag{A.52}$$

Recalling the already established bound for (A.52), we obtain

$$\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}}\operatorname{Var}\left(\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right)\leq \sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}} C\rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{1}{1+\delta}}\leq C\,r_k^{\frac{1}{1+\delta}}.$$

Hence, we set

$$V_{II,k}:=C\,r_k^{\frac{1}{1+\delta}}.$$

Having in mind that $r_{k_T}\leq r_k$ is fulfilled by construction, we take up on (A.51) to get

$$\begin{aligned}
&P\left(\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}}\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\right.\right.\\
&\qquad\left.\left.\cdot\left(\breve{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\breve{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|>\sum_{k=1}^{k_T}\lambda_k\right)\\
&\leq 2\sum_{k=1}^{k_T}D(k)D(k-1)\exp\left(-\frac{1}{2}\cdot\frac{\lambda_k^2}{V_{II,k}+\frac{\breve{M}\lambda_k}{3}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2 \sum_{k=1}^{k_T} \exp\left(2\log\left(D(k)\right) - \frac{1}{2} \cdot \frac{\lambda_k^2}{C_1\, r_k^{\frac{1}{1+\delta}} + \frac{4\, d_T^{\frac{(2m-3)\delta-4}{2m(4+3\delta)}}\, \lambda_k}{3}}\right) \\
&\leq 2 \sum_{k=1}^{k_T} \exp\left(2\log\left(D(k)\right) - \frac{1}{2} \cdot \frac{\lambda_k^2}{C_1\, r_k^{\frac{1}{1+\delta}} + C_2\, r_{k_T}^{\frac{1}{1+\delta}}\, \lambda}\right) \\
&\leq 2 \sum_{k=1}^{k_T} \exp\left(2\log\left(D(k)\right) - \frac{1}{2} \cdot \frac{\lambda_k^2}{C\, r_k^{\frac{1}{1+\delta}}}\right) \\
&\leq 2 \sum_{k=1}^{k_T} \exp\left(2\log\left(D(k)\right) - \bar{C}\, r_k^{-\frac{1}{1+\delta}}\, \lambda_k^2\right) \\
&\leq 2 \sum_{k=1}^{k_T} \exp\left(2\, \lambda_k^2\, \frac{\bar{C}}{4}\, r_k^{-\frac{1}{1+\delta}} - \bar{C}\, r_k^{-\frac{1}{1+\delta}}\, \lambda_k^2\right) \\
&= 2 \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2}\, \lambda_k^2\, r_k^{-\frac{1}{1+\delta}}\right) \\
&\leq 2 \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2}\, r_k^{\frac{1}{2(1+\delta)}}\, r_k^{-\frac{1}{1+\delta}}\right) \\
&\leq 2 \sum_{k\in\mathbb{N}} \exp\left(-C\, r^{-\frac{1}{2(1+\delta)}}\right) \\
&\underset{r\to 0}{\longrightarrow} 0
\end{aligned} \tag{A.53}
$$

with the help of (A.43).

Concerning term III of (A.45), we follow the same steps. We start again by bounding the product of weight and function:

$$
\begin{aligned}
\mathrm{III} = {} & P\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} \left|\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{27},\ \sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\ldots,T}} w_{t,T}\, f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) \leq d_T^{\frac{\delta}{4+3\delta}}\right) \\
& + P\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} \left|\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{27},\ \sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\ldots,T}} w_{t,T}\, f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) > d_T^{\frac{\delta}{4+3\delta}}\right) \\
\leq {} & P\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} \left|\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{27},\ \sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\ldots,T}} w_{t,T}\, f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) \leq d_T^{\frac{\delta}{4+3\delta}}\right) \\
& + P\left(\sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\ldots,T}} w_{t,T}\, f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right) > d_T^{\frac{\delta}{4+3\delta}}\right).
\end{aligned}
$$

Similarly, it suffices to look at

$$
\begin{aligned}
&P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} \left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{27}, \sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\ldots,T}} w_{t,T}\, f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\leq d_T^{\frac{\delta}{4+3\delta}}\right)\\
&= P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}}\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\breve{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\breve{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right| > \frac{\lambda}{27}\right).
\end{aligned}
\tag{A.54}
$$

Let V_{III} be the equivalent of $V_{II,k}$ from above. Then, we have again

$$
V_{III}\geq \sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} \operatorname{Var}\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\breve{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\breve{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right)
$$

and

$$
\begin{aligned}
&\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} \operatorname{Var}\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\breve{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\breve{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right)\\
&\leq \sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} C\rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{1}{1+\delta}}\\
&\leq Cr^{\frac{1}{1+\delta}}.
\end{aligned}
$$

Thus, we set

$$
V_{III} := Cr^{\frac{1}{1+\delta}}.
$$

With the very same $\breve{M}$ as before, we get for (A.54) using Bernstein's inequality

$$
\begin{aligned}
&P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}}\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\breve{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\breve{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right| > \frac{\lambda}{27}\right)\\
&\leq \sum_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} P\left(\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\breve{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\breve{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right| > \frac{\lambda}{27}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\,D^2(0)\exp\left(-\frac{1}{2}\cdot\frac{C_1\lambda^2}{V_{III}+\frac{\breve{M}C_2\lambda}{3}}\right)\\
&\leq 2\exp\left(2\log\left(D(0)\right)-\frac{1}{2}\cdot\frac{C_1\lambda^2}{V_{III}+\frac{\breve{M}C_2\lambda}{3}}\right)\\
&\leq 2\exp\left(2\log\left(D(0)\right)-\frac{1}{2}\cdot\frac{C_1\lambda^2}{C_2\,r^{\frac{1}{1+\delta}}+\frac{4\,d_T^{\frac{(2m-3)\delta-4}{2m(4+3\delta)}}\lambda}{3}}\right)\\
&\leq 2\exp\left(2\log\left(D(0)\right)-C\,r^{-\frac{1}{1+\delta}}\right)\\
&\underset{r\to 0}{\longrightarrow}0.
\end{aligned}
$$

Now we turn our attention to the remaining first summand in (A.45). Using Markov's inequality, we get

$$
P\left(2\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{27}\right)\leq\frac{54}{\lambda}\,E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|\right).
$$

Therefore, it is enough to check

$$
\lim_{r\to 0}\limsup_{T\to\infty}\,E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|\right)=0 \qquad \text{(A.55)}
$$

because it implies that term I vanishes as well. Before we start, once again we need some further notation. For $\underline{s}\in[-S,S]^d$ let

$$
L_{t,T}\left(\underline{s}\right):=\sum_{i\in H_t}w_{i,T}\,f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{i}{T}\right)\right)\qquad\text{and}\qquad L^0_{t,T}\left(\underline{s}\right):=\zeta_t L_{t,T}\left(\underline{s}\right),
$$

where $(\zeta_t)_{t=1}^{\mu_T}$ are i.i.d. Rademacher variables independent of $(\underline{\varepsilon}_t)_{t\in\mathbb{Z}}$. As we can see, the sequence $(L_{t,T}(\underline{s}))_{t=1}^{\mu_T}$ consists of independent random variables by the way we constructed it. Therefore, we can apply a standard symmetrization lemma as it can be found e.g. as Lemma 2.3.1 in van der Vaart and Wellner (2000) to get

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|\right)$$
$$= E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}\left(\underline{s}_1\right)-EL_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)+EL_{t,T}\left(\underline{s}_2\right)\right)\right|\right)$$
$$\le 2\,E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right)\right|\right). \tag{A.56}$$

The next step will be to verify that $\sum_{t=1}^{\mu_T} L^0_{t,T}$ has sub-Gaussian increments conditionally on $L_{1,T},\ldots,L_{\mu_T,T}$. This is the case since for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$ and $\eta>0$ we get by applying Hoeffding's inequality

$$P\left(\left|\sum_{t=1}^{\mu_T} L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right|>\hat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right)\eta\,\middle|\,L_{1,T},\ldots,L_{\mu_T,T}\right)$$
$$\le 2\exp\left(-\frac{\hat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right)^2\eta^2}{2\sum_{t=1}^{\mu_T}\left(L_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)\right)^2}\right)$$
$$=2\exp\left(-\frac{\eta^2}{2}\right) \tag{A.57}$$

with the random semimetric

$$\hat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right):=\left(\sum_{t=1}^{\mu_T}\left(L_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)\right)^2\right)^{1/2} \tag{A.58}$$

for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$. That is the reason why we want to verify (A.55) with help of a maximal inequality for sub-Gaussian processes. The following steps will be easier if we use a semimetric with slight differences compared to the one stated above. To obtain this new semimetric, we note that

$$\left(L_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)\right)^2\le\left(\left|L_{t,T}\left(\underline{s}_1\right)\right|+\left|L_{t,T}\left(\underline{s}_2\right)\right|\right)^{\frac{2+\delta}{3}}\left|L_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)\right|^{\frac{4-\delta}{3}}$$
$$\le 2^{\frac{2+\delta}{3}}\;\left|L_{t,T}\right|_\infty^{\frac{2+\delta}{3}}\left|L_{t,T}\right|_{\text{Lip}}^{\frac{4-\delta}{3}}\rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{4-\delta}{3}} \tag{A.59}$$

holds for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$. Besides, $|f|_{\text{Lip}}$ can be rewritten as

$$\sup_{\substack{\underline{s}_1\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\neq 0}} \frac{\left|\sum_{i\in H_t} w_{i,T}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|}{\rho\left(\underline{s}_1,\underline{s}_2\right)}.$$

By defining

$$Q_T := 2^{\frac{2+\delta}{6}}\left(\sum_{t=1}^{\mu_T}\left|L_{t,T}\right|_\infty^{\frac{2+\delta}{3}}\ \left|L_{t,T}\right|_{\text{Lip}}^{\frac{4-\delta}{3}}\right)^{1/2},$$

we get

$$\begin{aligned}
EQ_T &\leq 2^{\frac{2+\delta}{6}}\left(\sum_{t=1}^{\mu_T}E\left|L_{t,T}\right|_\infty^{\frac{2+\delta}{3}}\ \left|L_{t,T}\right|_{\text{Lip}}^{\frac{4-\delta}{3}}\right)^{1/2}\\
&\leq C\left(\sum_{t=1}^{\mu_T}\left(E\left|L_{t,T}\right|_\infty^{2+\delta}\right)^{1/3}\ \left(E\left|L_{t,T}\right|_{\text{Lip}}^{\frac{4-\delta}{2}}\right)^{2/3}\right)^{1/2}.
\end{aligned}\tag{A.60}$$

The next part will be to obtain bounds for the expectations. First, we have

$$\begin{aligned}
&\left\|L_{t,T}\right\|_\infty^{2+\delta}\\
&= E\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{i\in H_t} w_{i,T}\, f\left(\underline{s},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right)^{2+\delta}\\
&\leq E\left(\sup_{\underline{s}\in[-S,S]^d}\sum_{i\in H_t} w_{i,T}\left(\left|f\left(\underline{s},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right.\right.\\
&\qquad\qquad\left.\left.+\left|f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right)\right)^{2+\delta}\\
&\leq E\left(\sum_{i\in H_t} w_{i,T}\left(Sd\,g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right),\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)+\left|f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right)\right)^{2+\delta}\\
&= \left\|\sum_{i\in H_t} w_{i,T}\left(Sd\,g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right),\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)+\left|f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right)\right\|_{2+\delta}^{2+\delta}\\
&\leq \left(C_1\sum_{i\in H_t} d_T^{1/2}\left\|Sd\,g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right),\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)+\left|f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right\|_{2+\delta}\right)^{2+\delta}\\
&\leq \left(C_1\sum_{i\in H_t} d_T^{1/2}\left(\left\|Sd\,g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right),\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{2+\delta}\right.\right.\\
&\qquad\qquad\left.\left.+\left\|f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{2+\delta}\right)\right)^{2+\delta}
\end{aligned}$$

$$\leq C_1 \left(\kappa_T \, d_T^{1/2}\right)^{2+\delta}$$
$$= C_1 \, \kappa_T^{2+\delta} \, d_T^{\frac{2+\delta}{2}}.$$

The second one can be bounded via

$$\begin{aligned}
\|L_{t,T}\|_{\mathrm{Lip}}^{\frac{4-\delta}{2}} &= E\left(\sup_{\substack{\underline{s}_1 \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)\neq 0}} \frac{\left|\sum_{i\in H_t} w_{i,T}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|}{\rho\left(\underline{s}_1, \underline{s}_2\right)}\right)^{\frac{4-\delta}{2}} \\
&\leq E\left(\sum_{i\in H_t} w_{i,T} \, g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right), \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{4-\delta}{2}} \\
&\leq \left(\sum_{i\in H_t} w_{i,T} \left\|g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right), \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{\frac{4-\delta}{2}}\right)^{\frac{4-\delta}{2}} \\
&\leq C_2 \, \kappa_T^{\frac{4-\delta}{2}} \, d_T^{\frac{\frac{4-\delta}{2}}{2}}.
\end{aligned}$$

Together, we have for (A.60)

$$\begin{aligned}
EQ_T &\leq C\left(\sum_{t=1}^{\mu_T} \kappa_T^{(2+\delta)\frac{1}{3}+\frac{4-\delta}{2}\cdot\frac{2}{3}} \, d_T^{\frac{1}{2}\left((2+\delta)\frac{1}{3}+\frac{4-\delta}{2}\cdot\frac{2}{3}\right)}\right)^{1/2} \\
&= C\left(\mu_T \, \kappa_T^2 \, d_T\right)^{1/2} \\
&\leq C \, \kappa_T^{1/2} \\
&\leq C \, d_T^{-\frac{1}{4m}}.
\end{aligned} \tag{A.61}$$

Having the definition of Q_T in mind, it holds

$$\hat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) \leq Q_T \; \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{4-\delta}{6}} =: \breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right),$$

and $\breve{\rho}_T$ is again a random semimetric because it fulfills

$$\breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) = \breve{\rho}_T\left(\underline{s}_2, \underline{s}_1\right) \qquad \text{and} \qquad \breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) \geq 0,$$

and, as it holds $\frac{4-\delta}{6} \in (0,1)$, we get

$$\begin{aligned}
\breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) &\leq Q_T\left(\rho\left(\underline{s}_1, \underline{s}_3\right) + \rho\left(\underline{s}_3, \underline{s}_2\right)\right)^{\frac{4-\delta}{6}} \\
&\leq Q_T \rho\left(\underline{s}_1, \underline{s}_3\right)^{\frac{4-\delta}{6}} + Q_T \rho\left(\underline{s}_3, \underline{s}_2\right)^{\frac{4-\delta}{6}}
\end{aligned}$$

$$= \breve{\rho}_T\left(\underline{s}_1, \underline{s}_3\right) + \breve{\rho}_T\left(\underline{s}_3, \underline{s}_2\right).$$

Now we consider the conditional expectation

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right) - L^0_{t,T}\left(\underline{s}_2\right)\right)\right| \,\middle|\, L_{1,T},\ldots,L_{\mu_T,T}\right)$$

and make use of Corollary 2.2.8 of van der Vaart and Wellner (2000) to get

$$\begin{aligned}
&E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right) - L^0_{t,T}\left(\underline{s}_2\right)\right)\right| \,\middle|\, L_{1,T},\ldots,L_{\mu_T,T}\right)\\
&= E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \breve{\rho}_T(\underline{s}_1,\underline{s}_2)\le Q_T r_{k_T}^{\frac{4-\delta}{6}}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right) - L^0_{t,T}\left(\underline{s}_2\right)\right)\right| \,\middle|\, L_{1,T},\ldots,L_{\mu_T,T}\right)\\
&\le C_3\int_0^{Q_T r_{k_T}^{\frac{4-\delta}{6}}}\left(\log\left(D\left(u,[-S,S]^d,\breve{\rho}_T\right)\right)\right)^{1/2} du. \qquad \text{(A.62)}
\end{aligned}$$

Next, we want to establish an upper bound for the packing number and get

$$\begin{aligned}
&D\left(u,[-S,S]^d,\breve{\rho}_T\right)\\
&= \max\left\{\#\mathcal{T}_0 \,\middle|\, \mathcal{T}_0\subseteq[-S,S]^d,\ \breve{\rho}_T\left(\underline{s}_1,\underline{s}_2\right) > u\ \forall\, \underline{s}_1\neq\underline{s}_2\in\mathcal{T}_0\right\}\\
&= \max\left\{\#\mathcal{T}_0 \,\middle|\, \mathcal{T}_0\subseteq[-S,S]^d,\ \rho\left(\underline{s}_1,\underline{s}_2\right) > \left(\frac{u}{Q_T}\right)^{\frac{6}{4-\delta}}\ \forall\, \underline{s}_1\neq\underline{s}_2\in\mathcal{T}_0\right\}\\
&= D\left(\left(\frac{u}{Q_T}\right)^{\frac{6}{4-\delta}},[-S,S]^d,\rho\right)\\
&\le \left(\frac{2\,Sd}{\left(\frac{u}{Q_T}\right)^{\frac{6}{4-\delta}}} + 1\right)^d. \qquad \text{(A.63)}
\end{aligned}$$

Inserting this bound into (A.62) gives

$$
\begin{aligned}
&E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right)\right|\,\middle|\, L_{1,T},\dots,L_{\mu_T,T}\right)\\
&\le C_3\int_0^{Q_T r_{k_T}^{\frac{4-\delta}{6}}}\left(\log\left(\frac{2\,Sd}{\left(\frac{u}{Q_T}\right)^{\frac{6}{4-\delta}}}+1\right)^d\right)^{1/2}du\\
&= C_3\,Q_T\int_0^{r_{k_T}^{\frac{4-\delta}{6}}}\left(\log\left(\frac{2\,Sd}{u^{\frac{6}{4-\delta}}}+1\right)^d\right)^{1/2}du\\
&\le C_3\,Q_T\int_0^{r_{k_T}^{\frac{4-\delta}{6}}} u^{-\frac{3}{4-\delta}}\,du
\end{aligned}
\tag{A.64}
$$

using the fact that it holds $\log(x+1)\le x$ for $x>0$. Returning to (A.56) we get

$$
\begin{aligned}
&E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right)\right|\right)\\
&= EE\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right)\right|\,\middle|\, L_{1,T},\dots,L_{\mu_T,T}\right)\\
&\le C_3\,EQ_T\int_0^{r_{k_T}^{\frac{4-\delta}{6}}} u^{-\frac{3}{4-\delta}}\,du.
\end{aligned}
\tag{A.65}
$$

The integral can be calculated as follows:

$$
\int_0^{r_{k_T}^{\frac{4-\delta}{6}}} u^{-\frac{3}{4-\delta}}\,du=\left[\frac{4-\delta}{1-\delta}u^{\frac{1-\delta}{4-\delta}}\right]_0^{r_{k_T}^{\frac{4-\delta}{6}}}=C_4\,r_{k_T}^{\frac{1-\delta}{4-\delta}\cdot\frac{4-\delta}{6}}=C_4\,r_{k_T}^{\frac{1-\delta}{6}},
$$

and, finally, together with (A.61) and (A.46) we have

$$
\begin{aligned}
E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right)\right|\right)&\le C_3\,d_T^{-\frac{1}{4m}}\,r_{k_T}^{\frac{1-\delta}{6}}\\
&\le C_3\,d_T^{-\frac{1}{4m}}\,d_T^{\frac{2}{m(1-\delta)}\cdot\frac{1-\delta}{6}}
\end{aligned}
$$

$$= C_3\, d_T^{\frac{1}{12m}},$$

which tends to 0 as $T \to \infty$, and the proof is completed.

(b) As in part (a), we start by demanding both a lower and an upper bound for r_{k_T} of the following form:

$$d_T^{\frac{(m-1)(1+\delta)}{2m}} \le r_{k_T} \le d_T^{\frac{1}{m\delta}}. \tag{A.66}$$

Again, by our choice of m, we guarantee for the existence of a $k_T \in \mathbb{N}$ fulfilling the aforesaid requirements. This can be seen by transforming the following necessary inequality:

$$d_T^{\frac{(m-1)(1+\delta)}{2m}} < d_T^{\frac{1}{m\delta}}.$$

Equivalent to the equation above, we pursue looking at

$$\frac{(m-1)(1+\delta)}{2m} > \frac{1}{m\delta},$$

which, on the other hand, can be transformed into

$$(m-1)(1+\delta)\delta > 2.$$

Hence, we obtain

$$m > \frac{2+\delta(1+\delta)}{\delta(1+\delta)} = 1 + \frac{2}{\delta(1+\delta)}$$

as a requirement for m, which explains our choice.

At this point, we return to the three summands of equation (A.45). The treatment of the individual terms will be carried out analogously to part (a). On account of this, we start by establishing the necessary bounds for the application of Bernstein's inequality on term II. First, we constitute an upper bound for the variance of the inner sum of ν_T. In fact, the bound will be the same as in the previous part but the way it is calculated will be different. Consider $l := |i_1 - i_2|$. Then, we have

$$\begin{aligned}
&\operatorname{Var}\left(\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right) \\
&\leq \sum_{t=1}^{\mu_T} \sum_{i_1, i_2 \in H_t} w_{i_1,T}\, w_{i_2,T} \left| \operatorname{Cov}\left(\left(\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)\right.\right.\right. \\
&\qquad\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right), \\
&\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\quad + \sum_{t=1}^{\mu_T} \sum_{i_1, i_2 \in H_t} w_{i_1,T}\, w_{i_2,T} \left| \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right), \right.\right. \\
&\qquad\qquad \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right) \\
&\qquad\qquad\qquad \left.\left. - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M(l))}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M(l))}\left(\frac{i_2}{T}\right)\right)\right)\right)\right|
\end{aligned} \tag{A.67}$$

for the truncation parameter $M(l) := \lceil \min\{M, l/2\} \rceil$ as it is explained at length in equation (A.47). Next, we split up the covariances. As in part (a), we focus on the first and get again

$$\begin{aligned}
&\left| \operatorname{Cov}\left(\left(\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)\right.\right.\right. \\
&\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right), \\
&\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq \left| \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right), \right.\right. \\
&\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\quad + \left| \operatorname{Cov}\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right), \right.\right. \\
&\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|.
\end{aligned} \tag{A.68}$$

We concentrate once more on the first covariance of (A.68) because of similar behaviour. As seen in (a), we are in need of two alternative bounds for (A.68) in order to bound the variance of ν_T suitable for the use of Bernstein's inequality.

i) We obtain for the said covariance of (A.68)

$$
\begin{aligned}
&\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right.\\
&\qquad\qquad\left.\left.f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|\\
&\leq E\left|\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\right.\\
&\qquad\left.\cdot\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|\\
&\qquad+E\left|f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right|\\
&\qquad\qquad\cdot E\left|f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right|\\
&\leq CE\left|f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right| \qquad \text{(A.69)}
\end{aligned}
$$

due to the boundedness of our function f. Using the Lipschitz condition (2.11) in combination with Lemma 2.3, we get for the expectation stated in (A.69)

$$
E\left|f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right|\leq C\left\|\varepsilon_0\right\|_1 d\sum_{M>|j|\geq M(l)}\frac{B}{l(j)},
$$

which leads to

$$
\begin{aligned}
&\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right.\\
&\qquad\qquad\left.\left.f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|\\
&\leq C\sum_{M>|j|\geq M(l)}\frac{B}{l(j)}.
\end{aligned}
$$

ii) Considering the first covariance of (A.68), we get

$$
\begin{aligned}
&\left|\mathrm{Cov}\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right.\\
&\qquad\qquad\left.\left.f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\right. \\
&\qquad\qquad \left.\cdot\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\qquad + E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right| \\
&\qquad\qquad\qquad \cdot E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right| \qquad \text{(A.70)}
\end{aligned}$$

as written in (A.49). For the second summand of (A.70), we obtain again by using the Lipschitz condition in Assumption 2

$$\begin{aligned}
&E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right| \\
&\qquad\qquad \cdot E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right| \\
&\leq C\rho\left(\underline{s}_1, \underline{s}_2\right),
\end{aligned}$$

since f is bounded. We use the boundedness of f once more to get

$$\begin{aligned}
&E\left|\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\right. \\
&\qquad\qquad \left.\cdot\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\qquad + E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq CE\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right| \\
&\leq C\rho\left(\underline{s}_1, \underline{s}_2\right)
\end{aligned}$$

for the first summand of (A.70). Consequently, it holds for (A.70)

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right. \\
&\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq C\rho\left(\underline{s}_1, \underline{s}_2\right),
\end{aligned}$$

and the second version is completed.

Combining both bounds gives us

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\right.\right. \\
&\qquad\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq C \min\left\{\rho\left(\underline{s}_1, \underline{s}_2\right), \sum_{M>|j|\geq M(l)} \frac{B}{l(j)}\right\}
\end{aligned}$$

and thus for the covariance in (A.67)

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)\right.\right. \\
&\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right), \\
&\qquad\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right| \\
&\leq C \min\left\{\rho\left(\underline{s}_1, \underline{s}_2\right), \sum_{M>|j|\geq M(l)} \frac{B}{l(j)}\right\}
\end{aligned}$$

since the second covariance of (A.68) mimics the behavior of the first. Because of our choice of m, it holds $m > \frac{1+\delta}{\delta}$. This can be seen by examining the following inequality:

$$1 + \frac{2}{\delta(1+\delta)} - \frac{1+\delta}{\delta} > 0,$$

which is equal to

$$2 + \delta(1+\delta) - 1 - 2\delta - \delta^2 > 0.$$

Concentrating the left-hand side gives $1 - \delta > 0$, which is fulfilled for every $\delta \in (0, 1)$. This allows us to borrow the last steps of the calculation of the bound the variance of ν_T directly from part (a). Therefore, we can state

$$\operatorname{Var}\left(\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right) \leq C\rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{1}{1+\delta}}.$$

At this point, we go back to term II of equation (A.45). Using Bernstein's inequality and the notation we introduced in (A.44), we get

$$
\begin{aligned}
\mathrm{II} = P \Bigg(\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} \Bigg| \sum_{t=1}^{\mu_T} \sum_{i \in H_t} w_{i,T} & \\
\cdot \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \Bigg| > \sum_{k=1}^{k_T} \lambda_k \Bigg) & \\
\leq \sum_{k=1}^{k_T} \sum_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} P \Bigg(\sum_{t=1}^{\mu_T} \Bigg| \sum_{i \in H_t} w_{i,T} & \\
\cdot \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \Bigg| > \lambda_k \Bigg) & \\
\leq 2 \sum_{k=1}^{k_T} D(k) D(k-1) \exp\left(-\frac{1}{2} \cdot \frac{\lambda_k^2}{V_{II,k} + \frac{\breve{M}\lambda_k}{3}} \right). & \qquad \text{(A.71)}
\end{aligned}
$$

As in part (a), $\breve{M}$ denotes the upper bound for the first absolute moment of the sum over i, and $V_{II,k}$ stands for the upper bound of the accumulated variances depending on k. Consequently, the specification of these bounds is the next part. Starting with $\breve{M}$, this time the appurtenant constraint has the following form:

$$
\breve{M} \geq \left| \sum_{i \in H_t} w_{i,T} \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right|
$$

for all $\underline{s}_1, \underline{s}_2 \in [-S, S]^d$ and $t = 1, \ldots, \mu_T$. Changing this condition under the use of the function's bound C_f eliminates the influence of $\underline{s}_1$ and $\underline{s}_2$. Thus, we get

$$
\breve{M} \geq 4\, C \sum_{i \in H_t} d_T^{1/2}
$$

for $t = 1, \ldots, \mu_T$. Since the summands do not depend on t any more, we ask for

$$
\breve{M} \geq C\, \kappa_T\, d_T^{1/2}
$$

instead. Recalling the definition of κ_T, this turns into

$$
\breve{M} \geq C\, d_T^{-\frac{1}{2m}}\, d_T^{\frac{1}{2}} = C\, d_T^{\frac{m-1}{2m}}.
$$

Therefore, we define $\breve{M}$ via

$$
\breve{M} := C\, d_T^{\frac{m-1}{2m}}.
$$

This finishes the specification of the upper bound belonging to the first absolute moment of the sum over i. Hence, we can focus on the upper bound for the variance. Following Bernstein's inequality, we should have

$$V_{II,k} \geq \sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} \operatorname{Var}\left(\sum_{t=1}^{\mu_T} \sum_{i \in H_t} w_{i,T} \left(\breve{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \breve{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right).$$

We remember the bound we identified for (A.67) and get

$$\sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} \operatorname{Var}\left(\nu_T\left(\underline{s}_1, \underline{s}_2\right)\right) \leq \sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} C \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{1}{1+\delta}} \leq C\, r_k^{\frac{1}{1+\delta}}.$$

Thus, we choose

$$V_{II,k} := C\, r_k^{\frac{1}{1+\delta}}.$$

Now we go back to equation (A.70). Similarly to (A.53), we obtain

$$\begin{aligned}
\mathrm{II} &\leq 2 \sum_{k=1}^{k_T} \exp\left(2 \log\left(D(k)\right) - \frac{1}{2} \cdot \frac{\lambda_k^2}{C\, r_k^{\frac{1}{1+\delta}} + \frac{C\, d_T^{\frac{m-1}{2m}}\, \lambda_k}{3}}\right) \\
&\leq 2 \sum_{k=1}^{k_T} \exp\left(2 \log\left(D(k)\right) - \frac{1}{2} \cdot \frac{\lambda_k^2}{C\, r_k^{\frac{1}{1+\delta}}}\right) \\
&\leq 2 \sum_{k \in \mathbb{N}} \exp\left(-\frac{\overline{C}}{2}\, r^{-\frac{1}{2(1+\delta)}}\right) \\
&\underset{r \to 0}{\longrightarrow} 0.
\end{aligned}$$

Let V_{III} be the equivalent of $V_{II,k}$ from above. Then, we have again

$$V_{III} \geq \sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} \operatorname{Var}\left(\sum_{t=1}^{\mu_T} \sum_{i \in H_t} w_{i,T} \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right)$$

and

$$\begin{aligned}
&\sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} \operatorname{Var}\left(\sum_{t=1}^{\mu_T} \sum_{i \in H_t} w_{i,T} \left(\bar{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right) \\
&\leq \sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} C \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{1}{1+\delta}}
\end{aligned}$$

$$\leq C r^{\frac{1}{1+\delta}}.$$

Hence, we define

$$V_{III} := C r^{\frac{1}{1+\delta}}.$$

Using the same $\breve{M}$ as while treating term II, Bernstein's inequality applied to equation (A.54) gives us

$$\begin{aligned}
&P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}}\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|>\frac{\lambda}{27}\right)\\
&\leq \sum_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} P\left(\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|>\frac{\lambda}{27}\right)\\
&\leq 2\,D^2(0)\exp\left(-\frac{1}{2}\cdot\frac{C\lambda^2}{V_{III}+\frac{\breve{M}C\lambda}{3}}\right)\\
&\leq 2\exp\left(2\log\left(D(0)\right)-\frac{1}{2}\cdot\frac{C\lambda^2}{V_{III}+\frac{\breve{M}C\lambda}{3}}\right)\\
&\leq 2\exp\left(2\log\left(D(0)\right)-\frac{1}{2}\cdot\frac{C\lambda^2}{C\,r^{\frac{1}{1+\delta}}+\frac{C\,d_T^{\frac{m-1}{2m}}\,\lambda}{3}}\right)\\
&\leq 2\exp\left(2\log\left(D(0)\right)-\overline{C}\,r^{-\frac{1}{1+\delta}}\right)\\
&\underset{r\to 0}{\longrightarrow} 0.
\end{aligned}$$

Now term I is left. As in part (a), Markov's inequality allows us to deal with the expectation. Thus, we have to verify

$$\lim_{r\to 0}\limsup_{T\to\infty} E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|\right)=0 \tag{A.72}$$

as it implies the disappearance of term I. Again, as in the previous part, we define

$$L_{t,T}\left(\underline{s}\right) := \sum_{i\in H_t} w_{i,T}\, f\left(\underline{s},\underline{\widetilde{X}}_t^{(M)}\left(\frac{i}{T}\right)\right) \qquad \text{and} \qquad L_{t,T}^0\left(\underline{s}\right) := \zeta_t L_{t,T}\ \left(\underline{s}\right)$$

for $\underline{s}\in[-S,S]^d$ with $\left(\zeta_t\right)_t$ being i.i.d. Rademacher variables and independent of $\left(\underline{\varepsilon}_t\right)_t$. Since the sequence $\left(L_{t,T}(\underline{s})\right)_t$ contains only independent random variables by construction, the application of a standard symmetrization lemma as it is written in form of Lemma 2.3.1 in van der Vaart and Wellner (2000) is possible. Hence, we get

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|\right)\le 2\,E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right)\right|\right) \tag{A.73}$$

similar to equation (A.56). Now the verification of the fact that $\sum_{t=1}^{\mu_T} L^0_{t,T}$ has sub-Gaussian increments conditionally on $L_{1,T},\ldots,L_{\mu_T,T}$, $T\in\mathbb{N}$, falls into line. Hoeffding's inequality helps us because for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$ and $\eta>0$ we obtain

$$P\left(\left|\sum_{t=1}^{\mu_T}L^0_{t,T}\left(\underline{s}_1\right)-L^0_{t,T}\left(\underline{s}_2\right)\right|>\hat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right)\eta\,\middle|\,L_{1,T},\ldots,L_{\mu_T,T}\right)\le 2\exp\left(-\frac{\eta^2}{2}\right)$$

comparable to (A.57) using the same random semimetric as in (A.58), that is

$$\hat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right):=\left(\sum_{t=1}^{\mu_T}\left(L_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)\right)^2\right)^{1/2}$$

for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$. Thus, we can check for (A.72) with the use of a maximal inequality for sub-Gaussian processes again. To ease the following part, we use a slightly different semimetric as above. In order to get the said new semimetric, we notice that

$$\left(L_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)\right)^2\le 2^{\frac{2-\delta}{2}}\left|L_{t,T}\right|_\infty^{\frac{2-\delta}{2}}\left|L_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\;\rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{2+\delta}{2}} \tag{A.74}$$

holds for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$, comparable to (A.59). Next, set

$$Q_T:=2^{\frac{2-\delta}{4}}\left(\sum_{t=1}^{\mu_T}\left|L_{t,T}\right|_\infty^{\frac{2-\delta}{2}}\left|L_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\right)^{1/2}. \tag{A.75}$$

The subsequent step will be to bound $\left|L_{t,T}\right|_\infty^{\frac{2-\delta}{2}}$ and $\left|L_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}$. For the first, we get

$$\left|L_{t,T}\right|_\infty^{\frac{2-\delta}{2}}=\left(\sup_{\underline{s}\in[-S,S]^d}\left|\sum_{i\in H_t}w_{i,T}\,f\left(\underline{s},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right)^{\frac{2-\delta}{2}}\le C_1\,\kappa_T^{\frac{2-\delta}{2}}\,d_T^{\frac{2-\delta}{4}},$$

whereas the second receives its bound via

$$|L_{t,T}|_{\mathrm{Lip}}^{\frac{2+\delta}{2}} = \left(\sup_{\substack{\underline{s}_1 \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \neq 0}} \frac{\left| \sum_{i \in H_t} w_{i,T} \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right|}{\rho\left(\underline{s}_1, \underline{s}_2\right)} \right)^{\frac{2+\delta}{2}}$$
$$\leq \left(\sum_{i \in H_t} w_{i,T}\, g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right), \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{2+\delta}{2}}$$
$$\leq C_2\, d_T^{\frac{2+\delta}{4}} \left(\sum_{i \in H_t} g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right), \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{2+\delta}{2}} .$$

Combining both bounds, we obtain for (A.75)

$$Q_T \leq C_3 \left(\kappa_T^{\frac{2-\delta}{2}} d_T^{\frac{2-\delta}{4}} d_T^{\frac{2+\delta}{4}} \sum_{t=1}^{\mu_T} \left(\sum_{i \in H_t} g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right), \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{2+\delta}{2}} \right)^{1/2}$$
$$\leq C_3 \left(\kappa_T^{\frac{2-\delta}{2}} d_T \sum_{t=1}^{\mu_T} \left(\sum_{i \in H_t} g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right), \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{2+\delta}{2}} \right)^{1/2} . \tag{A.76}$$

Now we can define the new semimetric. Using the definition of Q_T, we have

$$\hat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) \leq Q_T\, \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{2+\delta}{4}} =: \breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right),$$

and $\breve{\rho}_T$ is again a random semimetric as it satisfies

$$\breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) = \breve{\rho}_T\left(\underline{s}_2, \underline{s}_1\right) \qquad \text{and} \qquad \breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) \geq 0$$

as well as

$$\breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) \leq Q_T \left(\rho\left(\underline{s}_1, \underline{s}_3\right) + \rho\left(\underline{s}_3, \underline{s}_2\right)\right)^{\frac{2+\delta}{4}} \leq \breve{\rho}_T\left(\underline{s}_1, \underline{s}_3\right) + \breve{\rho}_T\left(\underline{s}_3, \underline{s}_2\right)$$

because of $\frac{2+\delta}{4} \in (0,1)$. As in part (a), applying Corollary 2.2.8 of van der Vaart and Wellner (2000) we obtain for the conditional expectation the following:

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right)\right|\,\middle|\, L_{1,T},\ldots,L_{\mu_T,T}\right)$$

$$\le C\int_0^{Q_T r_{k_T}^{\frac{2+\delta}{4}}}\left(\log\left(D\left(u,[-S,S]^d,\breve{\rho}_T\right)\right)\right)^{1/2}du. \quad \text{(A.77)}$$

Again, we need an upper bound for the packing number. Comparable to the previous part, it holds

$$D\left(u,[-S,S]^d,\breve{\rho}_T\right)=D\left(\left(\frac{u}{Q_T}\right)^{\frac{4}{2+\delta}},[-S,S]^d,\rho\right)\le\left(\frac{2\,Sd}{\left(\frac{u}{Q_T}\right)^{\frac{4}{2+\delta}}}+1\right)^d.$$

Now we can insert this bound into (A.77) and get

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right)\right|\,\middle|\, L_{1,T},\ldots,L_{\mu_T,T}\right)\le C\,Q_T\int_0^{r_{k_T}^{\frac{2+\delta}{4}}}u^{-\frac{2}{2+\delta}}\,du$$

as seen similarly in equation (A.64). Going back to (A.73), it holds like in (A.65)

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right)\right|\right)\le C\,EQ_T\int_0^{r_{k_T}^{\frac{2+\delta}{4}}}u^{-\frac{2}{2+\delta}}\,du.$$

Next, we focus on EQ_T and get with the use of (A.76)

$$\begin{aligned}
EQ_T &\le C_3\left(\kappa_T^{\frac{2-\delta}{2}}\,d_T\sum_{t=1}^{\mu_T}\left\|\sum_{i\in H_t}g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right),\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{\frac{2+\delta}{2}}^{\frac{2+\delta}{2}}\right)^{1/2}\\
&\le C_3\left(\kappa_T^{\frac{2-\delta}{2}}\,d_T\sum_{t=1}^{\mu_T}\left(\sum_{i\in H_t}\left\|g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right),\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{\frac{2+\delta}{2}}\right)^{\frac{2+\delta}{2}}\right)^{1/2}\\
&\le C_3\left(\kappa_T^{\frac{2-\delta}{2}}\,d_T\,\mu_T\,\kappa_T^{\frac{2+\delta}{2}}\right)^{1/2}\\
&= C_3\left(\kappa_T^2\,d_T\,\mu_T\right)^{1/2}\\
&\le C_3\left(\kappa_T\right)^{1/2}
\end{aligned}$$

$$\leq C_3\, d_T^{-\frac{1}{4m}}. \tag{A.78}$$

For the integral it holds

$$\int_0^{r_{k_T}^{\frac{2+\delta}{4}}} u^{-\frac{2}{2+\delta}}\, du = \left[\frac{2+\delta}{\delta} u^{\frac{\delta}{2+\delta}}\right]_0^{r_{k_T}^{\frac{2+\delta}{4}}} = C_4\, r_{k_T}^{\frac{\delta}{2+\delta}\cdot\frac{2+\delta}{4}} = C_4\, r_{k_T}^{\delta/4}. \tag{A.79}$$

Finally, combining (A.78) and (A.79) and using the upper bound of r_{k_T} in (A.66) we have

$$\begin{aligned} E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0(\underline{s}_1)-L_{t,T}^0(\underline{s}_2)\right)\right|\right) &\leq C\, d_T^{-\frac{1}{4m}}\, r_{k_T}^{\delta/4} \\ &\leq C\, d_T^{-\frac{1}{4m}}\, d_T^{\frac{2}{m\delta}\cdot\frac{\delta}{4}} \\ &= C\, d_T^{\frac{1}{4m}}, \end{aligned}$$

which tends to 0 as $T\to\infty$, and the proof is completed.

□

Lastly, we use all of the already shown results to prove the FCLT presented in Theorem 2.19.

Proof of Theorem 2.19. Following Theorem 1.5.4 of van der Vaart and Wellner (2000), we need to show convergence of the fidis and asymptotical tightness in order to prove process convergence. However, Theorem 1.5.7 of van der Vaart and Wellner (2000) in combination with Lemma 2.17 allow us to show tightness as seen in Lemma 2.18. Theorem 2.15 gives the required convergence of the fidis and said Lemma 2.18 takes care of the tightness. Finally, the continuity of the sample path of the limiting process can be concluded with the help of Addendum 1.5.8. of van der Vaart and Wellner (2000). This finishes the proof. □

Therewith, the first stage of the proof journey is completed.

A.2. Proofs Belonging to Chapter 3

This section occupies itself with the proofs belonging to the bootstrap world, which we entered in Chapter 3. However, as we have already seen in said chapter, not all of our findings take place in the bootstrap world, but are needed for the demonstration of bootstrap-related results. The first half, namely subsections A.2.1 and A.2.2, deals with an unbounded function f, whereas the last two sections attend to a bounded one.

A.2.1. Proofs of Section 3.3

In this first subsection, we deal with the proofs connected with the establishment of the bootstrap CLT, whose proof will, eventually, conclude this subsection. On our way to this proof, we need additionally results, which will be presented and proven in this subsection as well. This first subsection

Nevertheless, we start with the demonstration of the proofs belonging to the vorangehenden results. Dazu, we mostly turn our back to the bootstrap world. The reason behind this is, as already explained, the need for additional general results originating in the real world. The first lemma we address is Lemma 3.4.

Proof of Lemma 3.4. We show the two parts one after the other, where the first will be treated extensively. The second, in turn, benefits strongly from the first, and thus can be cut short.

(i) As we have $t_1 \in \mathbb{N}$ and $t_2, r \in \mathbb{N}_0$ with $t_1 > t_2$, we can deduce

$$t_1 + r \geq t_1 > t_2 > 0.$$

Therefore, set $M := \left\lceil \frac{t_1 - t_2}{2} \right\rceil$. Now we are going to follow the proof of version (a) of Lemma 2.11 but modified for the covariance of products. Hence, we start by inserting the truncated version of the companion process as introduced in (2.13) with the above defined truncation parameter M and obtain using (3.3)

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\leq \left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right),\right.\right. \\
&\qquad\qquad \left.\left. \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\quad + \left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right),\right.\right. \\
&\qquad\qquad \left.\left. \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\quad + \left|\operatorname{Cov}\left(\bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{A.80}$$

Due to our choice for M, term III equals 0. As the terms I and II have a similar structure, we focus on term I and transfer the results to term II afterwards. We have

$$\begin{aligned}
\mathrm{I} = &\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right),\right.\right. \\
&\qquad \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right) \\
&\qquad\quad \left.\left. + \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big|\mathrm{Cov}\Big(\bar f\left(\underline{s},\underline{\widetilde X}_{t_1+r}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_1}(u)\right),\\
&\qquad\qquad \bar f\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)\left(\bar f\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)-\bar f_M\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)\right)\Big)\Big|\\
&\qquad\qquad + \Big|\mathrm{Cov}\Big(\bar f\left(\underline{s},\underline{\widetilde X}_{t_1+r}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_1}(u)\right),\\
&\qquad\qquad\qquad \bar f_M\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)\left(\bar f\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)-\bar f_M\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)\right)\Big)\Big|\\
&=: \mathrm{Ia}+\mathrm{Ib}.
\end{aligned}
\tag{A.81}
$$

For the same reasons as above, we limit ourselves to the investigation of the first subterm on (A.81) and obtain

$$
\begin{aligned}
\mathrm{Ia} &\leq \Big|E\Big(\bar f\left(\underline{s},\underline{\widetilde X}_{t_2+r}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_1}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)\\
&\qquad\qquad \cdot\left(\bar f\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)-\bar f_M\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)\right)\Big)\Big|\\
&\qquad\qquad + E\left|\bar f\left(\underline{s},\underline{\widetilde X}_{t_2+r}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_1}(u)\right)\right|\\
&\qquad\qquad\qquad \cdot E\left|\bar f\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)\left(\bar f\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)-\bar f_M\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)\right)\right|\\
&=: \mathrm{Iaa}+\mathrm{Iab}.
\end{aligned}
\tag{A.82}
$$

We continue with the first newly formed subterm and get using Hoelder's inequality

$$
\begin{aligned}
\mathrm{Iaa} &\leq E\left|\bar f\left(\underline{s},\underline{\widetilde X}_{t_1+r}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_1}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)\left(\bar f\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)-\bar f_M\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)\right)\right|\\
&\leq \left\|\bar f\left(\underline{s},\underline{\widetilde X}_{t_1+r}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_1}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)\right\|_{\frac{4+\delta}{3}}\\
&\qquad\qquad \cdot\left\|\bar f\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)-\bar f_M\left(\underline{s},\underline{\widetilde X}_{0}(u)\right)\right\|_{\frac{4+\delta}{1+\delta}}\\
&\leq C\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{\delta}{3+\delta}}
\end{aligned}
$$

since we have

$$
\left\|\bar f\left(\underline{s},\underline{\widetilde X}_{t_1+r}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_1}(u)\right)\bar f\left(\underline{s},\underline{\widetilde X}_{t_2}(u)\right)\right\|_{\frac{4+\delta}{3}} \leq C
$$

because of the finite absolute moments of order $4+\delta$ and

$$\begin{aligned}
&\left\|\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right\|_{\frac{4+\delta}{1+\delta}} \\
&= \left(E\left(\left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}(u)\right)\right|^{\frac{3(4+\delta)}{(3+\delta)(1+\delta)}}\right.\right. \\
&\qquad\qquad \left.\left. \cdot \left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}(u)\right)\right|^{\frac{\delta(4+\delta)}{(3+\delta)(1+\delta)}}\right)\right)^{\frac{1+\delta}{4+\delta}} \\
&\le \left(E\left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}(u)\right)\right|^{4+\delta}\right)^{\frac{3}{12+7\delta+\delta^2}} \\
&\qquad\qquad \cdot \left(E\left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}(u)\right)\right|\right)^{\frac{\delta}{(3+\delta)}} \\
&\le C\left(\sum_{|j|\ge M} \frac{B}{l(j)}\right)^{\frac{\delta}{3+\delta}}.
\end{aligned}$$

Now we turn to Iab of equation (A.82). We can bound the first factor of the product due to the moment condition and get

$$E\left|\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right)\right| \le C$$

for some finite constant $C > 0$. The second factor of Iab can be treated as follows:

$$\begin{aligned}
&E\left|\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right)\right| \\
&\le \left\|\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right\|_{4+\delta}\left\|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f_M\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right\|_{\frac{4+\delta}{3+\delta}} \\
&\le C\left(E\left(\left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f_M\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right|^{\frac{(4+\delta)}{(3+\delta)^2}}\right.\right. \\
&\qquad\qquad \left.\left. \cdot \left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f_M\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right|^{\frac{(2+\delta)(4+\delta)}{(3+\delta)^2}}\right)\right)^{\frac{3+\delta}{4+\delta}} \\
&\le C\left(\left(E\left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f_M\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right|^{4+\delta}\right)^{\frac{1}{(3+\delta)(4+\delta)}}\right. \\
&\qquad\qquad \left. \cdot \left(E\left|f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right) - f_M\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right|\right)^{\frac{2+\delta}{3+\delta}}\right) \\
&\le C\left(\sum_{|j|\ge M} \frac{B}{l(j)}\right)^{\frac{2+\delta}{3+\delta}}.
\end{aligned}$$

This gives for Ia eventually

$$\mathrm{Ia} \leq C\left(\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{\delta}{3+\delta}}+\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{2+\delta}{3+\delta}}\right).$$

In complete analogy, we get the same result as upper bound for Ib, which gives in conclusion

$$\mathrm{I} \leq C\left(\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{\delta}{3+\delta}}+\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{2+\delta}{3+\delta}}\right).$$

Moreover, as explained earlier, the right-hand side above also acts as upper bound for term II of equation (A.80). To sum it up, regarding all bounds we established above, it holds

$$\begin{aligned}&\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+t_2}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\&\qquad\leq C\left(\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{\delta}{3+\delta}}+\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{\frac{2+\delta}{3+\delta}}\right).\end{aligned}$$

To finish the proof, we make use of equation (3.1) in Assumption 5 and get

$$\begin{aligned}&\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+t_2}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\&\leq C\left(\left(\sum_{|j|\geq M}\frac{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}B}{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}l(j)}\right)^{\frac{\delta}{3+\delta}}+\left(\sum_{|j|\geq M}\frac{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}B}{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}l(j)}\right)^{\frac{2+\delta}{3+\delta}}\right)\\&\leq C\left(\left(\frac{1}{(t_1-t_2)^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}}\sum_{|j|\geq M}\frac{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}B}{l(j)}\right)^{\frac{\delta}{3+\delta}}\right.\\&\qquad\left.+\left(\frac{1}{(t_1-t_2)^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}}\sum_{|j|\geq M}\frac{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}}B}{l(j)}\right)^{\frac{2+\delta}{3+\delta}}\right)\\&\leq C\left(\frac{1}{(t_1-t_2)^{1+\tilde{\delta}}}+\frac{1}{(t_1-t_2)^{\frac{(1+\tilde{\delta})(2+\delta)}{\delta}}}\right)\\&\leq\frac{C_{Cov,2i}}{(t_1-t_2)^{1+\tilde{\delta}}}\end{aligned}$$

for some positive constant $C_{Cov,2i} < \infty$. This finishes the first part.

(ii) Because of $t_1, t_2 \in \mathbb{N}$ fulfilling $t_1 < t_2$, it holds

$$t_1 + t_2 > t_2 > t_1 > 0.$$

Now we distinguish between two different cases, namely

$$\text{a) } t_1 > \frac{t_2}{2} \quad \text{and} \quad \text{b) } t_1 \leq \frac{t_2}{2},$$

which we will examine one-by-one.

a) Analogously to the first part, we work with the truncated versions of the companion process. However, in this case, we set $M := \left\lceil \frac{t_2 - t_1}{2} \right\rceil$ as truncation parameter. Because the subsequent lines in the first part of this proof do not depend on the choice for M, we can directly move on with the following upper bound for the term in question

$$C\left(\left(\sum_{|j|\geq M} \frac{B}{l(j)}\right)^{\frac{\delta}{3+\delta}} + \left(\sum_{|j|\geq M} \frac{B}{l(j)}\right)^{\frac{2+\delta}{3+\delta}}\right).$$

This bound can be rewritten as

$$C\left(\left(\sum_{|j|\geq M} \frac{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}} B}{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}} l(j)}\right)^{\frac{\delta}{3+\delta}} + \left(\sum_{|j|\geq M} \frac{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}} B}{|j|^{\frac{(1+\tilde{\delta})(3+\delta)}{\delta}} l(j)}\right)^{\frac{2+\delta}{3+\delta}}\right). \tag{A.83}$$

Using Assumption 5 anew, we obtain the following upper bound for equation (A.83):

$$C\left(\frac{1}{(t_2 - t_1)^{1+\tilde{\delta}}} + \frac{1}{(t_2 - t_1)^{\frac{(1+\tilde{\delta})(2+\delta)}{\delta}}}\right) \leq \frac{C_{Cov,2ii1}}{(t_2 - t_1)^{1+\tilde{\delta}}}$$

with $C_{Cov,2ii1}$ being a finite positive constant.

b) Again, we rely on the truncated versions of the companion processes. In contrast to the previous cases, the truncation parameter has a different building type. This time, we consider $M := \lceil t_1/2 \rceil$. Following the same argumentation as in case (1), we proceed immediately with the upper bound for equation (A.83). In this case, we get

$$C\left(\frac{1}{t_1^{1+\tilde{\delta}}} + \frac{1}{t_1^{\frac{(1+\tilde{\delta})(2+\delta)}{\delta}}}\right) \leq \frac{C_{Cov,2ii2}}{t_1^{1+\tilde{\delta}}}$$

for some finite constant $C_{Cov,2ii2} > 0$.

Now we merge the two newly established constants $C_{Cov,2ii1}$ and $C_{Cov,2ii2}$ and define

$$C_{Cov,2ii} := \max\left\{C_{Cov,2ii1}, C_{Cov,2ii2}\right\}.$$

This finishes part (ii) and thus the whole proof.

□

After having established a bound for the covariance of products, we move on to the first auxiliary results, which cannot be found in the main part. At first, we aim to establish bounds for sums of expectations dealing with products. The following lemma illustrates three different types of these sums:

Lemma A.1.
Suppose Assumptions 5 and 6 hold true. Then, we have for all $\underline{s} \in [-S, S]^d$, $t_1, t_2 \in \mathbb{Z}$ *and* $u \in [0, 1]$

(i)

$$\frac{1}{(2TD_T+1)^2} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right) \leq \frac{C_{sum,1}}{TD_T},$$

(ii)

$$\frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right) \leq \frac{C_{sum,1}}{TD_T}$$

and

(iii)

$$\frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \sum_{m=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+m}(u)\right)\right) \leq \frac{C_{sum,3}}{TD_T},$$

where $\bar{f}$ *is defined in (2.15) and* $C_{sum,1}, C_{sum,2}$ *and* $C_{sum,3}$ *are each positive finite constants not depending either on* t_1 *and* t_2 *or on* $\underline{s}$ *and* u.

Proof. (i) Due to stationarity, we get

$$\begin{aligned}
&\sum_{l=-TD_T}^{TD_T}\sum_{k=-TD_T}^{TD_T}\left|E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+k}(u)\right)\right)\right| \\
&=\sum_{l=-TD_T}^{TD_T}\sum_{k=-TD_T}^{TD_T}\left|E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1-t_2+l-k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&=\sum_{l=-TD_T}^{TD_T}\sum_{k=-TD_T}^{TD_T}\left|\mathrm{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{t_1-t_2+l-k}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&=\sum_{\substack{l=-2TD_T\\ l\neq t_2-t_1}}^{2TD_T}(2TD_T+1-|l|)\left|\mathrm{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{t_1-t_2+l}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\qquad\qquad+(2TD_T+1-|t_2-t_1|)\left|\mathrm{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&=:\mathrm{I}+\mathrm{II}.
\end{aligned}$$

At this point, we continue by treating the accrued terms I and II singly. Since the covariance is finite, we obtain $\mathrm{II}\leq C_{sum,1a}TD_T$ for some positive constant $C_{sum,1a}<\infty$ immediately. Thus, we can go on and make use of version (a) of Lemma 2.11 to bound the covariances occurring in term I by $\frac{C_{Cov}}{|t_1-t_2+l|^{1+\tilde{\delta}}}$, respectively. This leads to

$$\begin{aligned}
\mathrm{I}&\leq\sum_{\substack{l=-2TD_T\\ l\neq t_2-t_1}}^{2TD_T}(2TD_T+1-|l|)\frac{C_{Cov}}{|t_1-t_2+l|^{1+\tilde{\delta}}} \\
&\leq C\,TD_T\sum_{\substack{l=-2TD_T\\ t\neq t_2-t_1}}^{2TD_T}\frac{1}{|t_1-t_2+l|^{1+\tilde{\delta}}} \\
&\leq C_{sum,1b}TD_T
\end{aligned}$$

for finite constant $C_{sum,1b}>0$, which implies together with the previous result

$$\frac{1}{(2TD_T+1)^2}\sum_{l=-TD_T}^{TD_T}\sum_{k=-TD_T}^{TD_T}\left|E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+k}(u)\right)\right)\right|\leq\frac{C_{sum,1}}{TD_T}$$

with $C_{sum,1}$ being a finite positive constant.

(ii) W.l.o.g. suppose $t_1 \geq t_2$. Moreover, set $v := t_1 - t_2$. Because of stationarity, we have

$$\begin{aligned}
&\Bigg| \frac{1}{(2\,TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right)\Bigg| \\
&= \Bigg| \frac{1}{(2\,TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\Bigg| \\
&\leq \Bigg| \frac{1}{(2\,TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l=-TD_T\\ l\neq r}}^{TD_T} \sum_{\substack{k=-TD_T\\ k\neq r}}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\Bigg| \\
&\quad + \left| \frac{1}{(2\,TD_T+1)^3} \sum_{\substack{l,r=-TD_T\\ l=r}}^{TD_T} \sum_{\substack{k=-TD_T\\ k\neq r}}^{TD_T} E\left(\bar{f}^2\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\right| \\
&\quad + \left| \frac{1}{(2\,TD_T+1)^3} \sum_{\substack{r,k=-TD_T\\ r=k}}^{TD_T} \sum_{l=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}^2\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

As f has finite $(4+\delta)$-th absolute moments, we can bound term II via

$$\mathrm{II} \leq \frac{C}{(2\,TD_T+1)^3}\,(TD_T+1)\,TD_T \leq \frac{C_{sum,2a}}{TD_T}$$

and term III, in complete analogy, by $\frac{C_{sum,2b}}{TD_T}$ for positive finite constants $C_{sum,2a}$ and $C_{sum,2b}$, respectively. Now we have a look at the remaining term I. It holds

$$
\begin{aligned}
\mathrm{I} \leq & \left| \frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l=-TD_T\\ l<r}}^{TD_T} \sum_{\substack{k=-TD_T\\ k<r}}^{TD_T} \right. \\
& \left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right| \\
& + \left| \frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l=-TD_T\\ l<r}}^{TD_T} \sum_{\substack{k=-TD_T\\ k>r}}^{TD_T} \right. \\
& \left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right| \\
& + \left| \frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l=-TD_T\\ l>r}}^{TD_T} \sum_{\substack{k=-TD_T\\ k<r}}^{TD_T} \right. \\
& \left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right| \\
& + \left| \frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l=-TD_T\\ l>r}}^{TD_T} \sum_{\substack{k=-TD_T\\ k>r}}^{TD_T} \right. \\
& \left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right| \\
=: & \ \mathrm{Ia} + \mathrm{Ib} + \mathrm{Ic} + \mathrm{Id}.
\end{aligned}
$$

First, we look at Ia. Since $v = t_1 - t_2$ and $t_1 \geq t_2$, we can distinguish two different cases, that is $v > 0$ and $v = 0$. In every case different relations between the indices occur, that are for $v > 0$

a) $v + r > v + l > r > k$,
b) $v + r > r > v + l > k$,
c) $v + r > r = v + l > k$,
d) $v + r > r > k > v + l$,
e) $v + r > r > v + l = k$

and for $v = 0$

a) $r > k > l$,
b) $r > l > k$,
c) $r > l = k$,

respectively. It is noticeable that the cases for $v = 0$ can be found hidden in those for $v > 0$ in which v has no influence on the order between r, l and k. Therefore, it suffice to look at the first five cases having in mind that the results apply to the last three as well. In every case in which equality between two indices appears, one sum drops out. Consequently, we get $\frac{C_{sum,2c}}{TD_T}$ as an upper bound for these cases for $0 < C_{sum,2c} < \infty$. This depletes the number of cases to analyse. Having a closer look at the remaining ones, we see that the principle of relation between the indices stays the same and only their order varies. So it is enough to look at one special case, and the ratio for the others results in the same way. To this end, we choose the first peculiarity of $v > 0$, to wit $v + r > v + l > r > k$. Then, we have

$$\begin{aligned}
&\Bigg| \frac{1}{(2\,TD_T+1)^3} \sum_{k=-TD_T}^{TD_T} \sum_{r=k+1}^{TD_T} \sum_{l=r-v+1}^{r+v} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\Bigg| \\
&\leq \frac{1}{(2\,TD_T+1)^3} \sum_{k=-TD_T}^{TD_T} \sum_{r=k+1}^{TD_T} \sum_{l=r-v+1}^{r+v} \\
&\qquad \left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\right| \\
&\qquad + \frac{1}{(2\,TD_T+1)^3} \sum_{k=-TD_T}^{TD_T} \sum_{r=k+1}^{TD_T} \sum_{l=r-v+1}^{r+v} \\
&\qquad \left|E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right)\right)\right| \left|E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\right| \\
&=: \breve{\mathrm{I}} + \breve{\mathrm{II}}.
\end{aligned}$$

The second term can be bounded via

$$\begin{aligned}
\breve{\mathrm{II}} &= \frac{1}{(2\,TD_T+1)^3} \sum_{k=-TD_T}^{TD_T} \sum_{r=k+1}^{TD_T} \left|E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\right| \\
&\qquad \cdot \sum_{l=r-v+1}^{r+v} \left|E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right)\right)\right| \\
&\leq \frac{C}{(2\,TD_T+1)^3} \sum_{k=-TD_T}^{TD_T} \sum_{r=k+1}^{TD_T} \left|\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right)\right| v \\
&\leq \frac{C_{sum,2d}}{(TD_T)^2}
\end{aligned}$$

for a positive constant $C_{sum,2d} < \infty$, whereas for the first term, we have to make use of the first part of Lemma 3.4 in order bound the covariance similar to the previous part of this proof. This leads to

$$
\begin{aligned}
\breve{\mathrm{I}} &\leq \frac{C}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=r-v+1}^{r+v} \frac{1}{(v+l-r)^{1+\tilde{\delta}}} \\
&\leq \frac{C}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=1}^{2v} \frac{1}{l^{1+\tilde{\delta}}} \\
&\leq \frac{C_{sum,2e}}{TD_T}
\end{aligned}
$$

with some constant $0 < C_{sum,2e} < \infty$. To conclude, we have a look at the bounds we got and see that the dominant denominator is $(TD_T)^{-1}$, which closes part (ii) as the same result can be shown in the vary same manner for Ib, Ic and Id.

(iii) Remembering from the previous part that the cases where t_1 equals t_2 are hidden in those for $t_1 > t_2$, we assume for our calculations $t_1 > t_2$ to lessen the cases to look at. Setting again $v := t_1 - t_2$, we have due to stationarity

$$
\begin{aligned}
&\Bigg| \frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \sum_{m=-TD_T}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+m}(u)\right)\right) \Bigg| \\
&= \Bigg| \frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \sum_{m=-TD_T}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{m}(u)\right)\right) \Bigg|
\end{aligned}
$$

$$\leq \left| \frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{\substack{k=-TD_T \\ k\neq r}}^{TD_T} \sum_{\substack{m=-TD_T \\ m\neq l}}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{m}(u)\right)\right) \right|$$

$$+ \left| \frac{1}{(2TD_T+1)^4} \sum_{\substack{r,k=-TD_T \\ r=k}}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{\substack{m=-TD_T \\ m\neq l}}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{m}(u)\right)\right) \right|$$

$$+ \left| \frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l,m=-TD_T \\ l=m}}^{TD_T} \sum_{\substack{k=-TD_T \\ k\neq r}}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{m}(u)\right)\right) \right|$$

$$+ \left| \frac{1}{(2TD_T+1)^4} \sum_{\substack{r,k=-TD_T \\ r=k}}^{TD_T} \sum_{\substack{l,m=-TD_T \\ l=m}}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{m}(u)\right)\right) \right|$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

Owing to the finite absolute moments of order $4+\delta$ the function f has, it holds

$$\mathrm{II} \leq \frac{C}{(2TD_T+1)^4} (2TD_T+1)^2 \, TD_T \leq \frac{C_{sum,3a}}{TD_T}$$

as well as

$$\mathrm{III} \leq \frac{C_{sum,3b}}{TD_T} \qquad \text{and} \qquad \mathrm{IV} \leq \frac{C_{sum,3c}}{(TD_T)^2}$$

using finite positive constants $C_{sum,3a}, C_{sum,3b}$ and $C_{sum,3c}$. As seen in part (ii), the next step is to split the remaining term I up further. By doing so, we obtain

$$\left| \frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{\substack{k=-TD_T \\ k\neq r}}^{TD_T} \sum_{\substack{m=-TD_T \\ m\neq l}}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{m}(u)\right)\right) \right|$$

$$
\begin{aligned}
\leq & \left| \frac{1}{(2\,TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=r+1}^{TD_T} \sum_{m=l+1}^{TD_T} \right. \\
& \left. E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{m}(u)\right)\right)\right| \\
& + \left| \frac{1}{(2\,TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=r+1}^{TD_T} \sum_{m=-TD_T}^{l-1} \right. \\
& \left. E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{m}(u)\right)\right)\right| \\
& + \left| \frac{1}{(2\,TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{r-1} \sum_{m=-TD_T}^{l-1} \right. \\
& \left. E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{m}(u)\right)\right)\right| \\
& + \left| \frac{1}{(2\,TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{r-1} \sum_{m=l+1}^{TD_T} \right. \\
& \left. E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{m}(u)\right)\right)\right| \\
=: & \ \mathrm{Ia} + \mathrm{Ib} + \mathrm{Ic} + \mathrm{Id}.
\end{aligned}
$$

Again, we have a look at Ia first, since the then-obtained result stays true for Ib, Ic and Id as well. Also similar to the previous part, we distinguish between different cases as there are:

a) $v+k > v+r > m > l$
b) $v+k > v+r = m > l$
c) $v+k > m > l > v+r$
d) $v+k = m > l > v+r$
e) $v+k = m > l = v+r$
f) $v+k > m > v+r > l$
g) $v+k > m > v+r = l$
h) $v+k = m > v+r > l$
i) $m > l > v+k > v+r$
j) $m > l = v+k > v+r$
k) $m > v+k > v+r > l$
l) $m > v+k > v+r = l$
m) $m > v+k > l > v+r$.

As before, in every case with equity at least one sum drops out, and overall, the bound we get for these cases is $\frac{C_{sum,3d}}{TD_T}$ using the finite constant $C_{sum,3d} > 0$. Now we pick again one remaining case, and the others follow similarly. We choose the first, that is $v+k > v+r > m > l$, and get

$$\begin{aligned}
&\Bigg|\frac{1}{(2TD_T+1)^4}\sum_{l=-TD_T}^{TD_T}\sum_{m=l+r}^{TD_T}\sum_{r=m-v+1}^{TD_T}\sum_{k=r+1}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{m}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{l}(u)\right)\right)\Bigg| \\
&\le \Bigg|\frac{1}{(2TD_T+1)^4}\sum_{l=-TD_T}^{TD_T}\sum_{m=l+r}^{TD_T}\sum_{r=m-v+1}^{TD_T}\sum_{k=r+1}^{TD_T} \\
&\qquad \operatorname{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+r}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{m}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{l}(u)\right)\right)\Bigg| \\
&\qquad + \Bigg|\frac{1}{(2TD_T+1)^4}\sum_{l=-TD_T}^{TD_T}\sum_{m=l+r}^{TD_T}\sum_{r=m-v+1}^{TD_T}\sum_{k=r+1}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v+r}(u)\right)\right)E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{m}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{l}(u)\right)\right)\Bigg| \\
&=: \breve{\mathrm{I}} + \breve{\mathrm{II}}.
\end{aligned}$$

As in part (ii), term $\breve{\mathrm{II}}$ can be bounded by $\frac{C_{sum,3e}}{(TD_T)^2}$, where $C_{sum,3e}$ is a finite positive constant, and part (i) of Lemma 3.4 helps us to get for term $\breve{\mathrm{I}}$

$$\breve{\mathrm{I}} \le \frac{C}{(2TD_T+1)^2}\sum_{m=-TD_T}^{TD_T}\sum_{r=m-v+1}^{TD_T}\frac{1}{(v+r-m)^{1+\tilde{\delta}}} \le \frac{C_{sum,3f}}{TD_T},$$

which gives a concluding upper bound $\frac{C_{sum,3}}{TD_T}$ for some constant $0 < C_{sum,3} < \infty$ and finishes the proof.

□

We remain in the examination of products of the function f. However, we change the object of observation as we focus on the arguments from henceforth:

Lemma A.2.
Under Assumption 5 we have for all $t_1, t_2 \in \{1, \ldots, T\}$ and $\underline{s} \in [-S, S]^d$

$$f\left(\underline{s},\underline{X}_{t_1,T}\right)f\left(\underline{s},\underline{X}_{t_2,T}\right) = f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)f\left(\underline{s},\underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right) + \mathcal{O}_P\left(T^{-\frac{3+\delta}{2(4+\delta)}}\right).$$

Here, the $\mathcal{O}_P$-term does not depend on the choices for t_1 and t_2 or $\underline{s}$.

Proof. We want to make use of the closeness between the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ and its companion process $\left(\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)_{t\in\mathbb{Z}}$. So the first step is to rewrite the difference between the two products:

$$
\begin{aligned}
&E\left|f\left(\underline{s},\underline{X}_{t_1,T}\right)f\left(\underline{s},\underline{X}_{t_2,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)f\left(\underline{s},\underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)\right|\\
&\leq E\left|\left(f\left(\underline{s},\underline{X}_{t_1,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right)f\left(\underline{s},\underline{X}_{t_2,T}\right)\right|\\
&\qquad\qquad+E\left|f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\left(f\left(\underline{s},\underline{X}_{t_2,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)\right)\right|\\
&=:\mathrm{I}+\mathrm{II}.
\end{aligned}
$$

Starting with I, we get using Lemma 2.6

$$
\begin{aligned}
\mathrm{I}&\leq\left\|f\left(\underline{s},\underline{X}_{t_1,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right\|_{\frac{4+\delta}{3+\delta}}\left\|f\left(\underline{s},\underline{X}_{t_2,T}\right)\right\|_{4+\delta}\\
&\leq C\left(E\left(\left|f\left(\underline{s},\underline{X}_{t_1,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right|^{1/2}\right.\right.\\
&\qquad\left.\left.\cdot\left|f\left(\underline{s},\underline{X}_{t_1,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right|^{\frac{4+\delta}{3+\delta}-\frac{1}{2}}\right)\right)^{\frac{3+\delta}{4+\delta}}\\
&\leq C\left(\left(E\left|f\left(\underline{s},\underline{X}_{t_1,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right|\right)^{\frac{3+\delta}{2(4+\delta)}}\right.\\
&\qquad\left.\cdot\left(E\left|f\left(\underline{s},\underline{X}_{t_1,T}\right)-f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right|^{\frac{2(5+\delta)}{6+\delta}}\right)^{\frac{3+\delta}{2(4+\delta)}}\right)\\
&\leq\frac{C_1}{T^{\frac{3+\delta}{2(4+\delta)}}},
\end{aligned}
$$

and hence, we obtain $\mathrm{II}\leq\frac{C_2}{T^{\frac{3+\delta}{2(4+\delta)}}}$ in complete analogy. In conclusion, this gives

$$
f\left(\underline{s},\underline{X}_{t_1,T}\right)f\left(\underline{s},\underline{X}_{t_2,T}\right)=f\left(\underline{s},\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)f\left(\underline{s},\underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)+\mathcal{O}_P\left(T^{-\frac{3+\delta}{2(4+\delta)}}\right),
$$

which is the striven for result. □

We continue with a new assistant lemma, but at this point, we make on step towards the bootstrap scenario because we take the window parameter D_T into consideration. This is visible in the following lemma, as said parameter is decisive in the occurring $\mathcal{O}_P$-terms:

Lemma A.3.
Suppose Assumptions 5 and 6 are fulfilled. Then, for all $h, k \in \{1, \dots, T\}$, $-TD_T \leq r, l \leq TD_T$ and $\underline{s} \in [-S, S]^d$ it holds

(i)

$$f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) = f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) + \mathcal{O}_P\left(D_T\right)$$

and

(ii)

$$f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right)$$
$$= f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right) + \mathcal{O}_P\left(D_T^{\frac{3+\delta}{2(4+\delta)}}\right).$$

Both $\mathcal{O}_P$-terms are independent of the choices for h, k, r, l and $\underline{s}$.

Proof. To show both the first and the second part, we want to benefit from the fact that while changing the argument but holding on to the index the difference between the two processes can be bounded nicely as it is stated in Lemma 2.3.

(i) Applying the second part of Lemma 2.3 after using the Lipschitz condition (2.11), we get part (i) straightforwardly, as it holds

$$\begin{aligned}
&E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right| \\
&\leq C_{Lip}\left\|\underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right) - \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right\|_1 \\
&\leq C\,\frac{|r|}{T} \\
&\leq C\,D_T
\end{aligned}$$

for $-TD_T \leq r \leq TD_T$.

(ii) To prove part (ii), the first step is to create the kind of difference Lemma 2.3 can be used on like in the proof of part (i). To this end, we have

$$\begin{aligned}
&E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right)\right| \\
&\leq E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right| \\
&\quad + E\left|\left(f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right| \\
&\quad + E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right)\left(f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right)\right|
\end{aligned}$$

$=: \mathrm{I} + \mathrm{II} + \mathrm{III}.$

A closer look at terms II and III shows that they are similarly built. Hence, we focus on II and get the same result for III. Using Hoelder's inequality, we get

$$\mathrm{II} \le \left\| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right\|_{\frac{4+\delta}{3+\delta}} \left\| f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right\|_{4+\delta}.$$

As f has uniformly finite $(4+\delta)$-th absolute moments, we can bound the last factor from above by a constant. Hence, we concentrate on the first factor. We aim to apply Hoelder's inequality for a second time in order to obtain a difference without exponent. Therefore, we rewrite the term firstly and get

$$\begin{aligned}
&\left\| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right\|_{\frac{4+\delta}{3+\delta}} \\
&= \left(E\left(\left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right|^{\frac{1}{2}} \right.\right. \\
&\qquad\qquad \left.\left. \cdot \left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right|^{\frac{4+\delta}{3+\delta}-\frac{1}{2}} \right)\right)^{\frac{3+\delta}{4+\delta}}. \qquad \text{(A.84)}
\end{aligned}$$

Now we can apply Hoelder's inequality to the right-hand side of equation (A.84) and obtain

$$\begin{aligned}
\mathrm{II} \le C &\left(\left(E\left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right| \right)^{\frac{3+\delta}{2(4+\delta)}} \right. \\
&\qquad \left. \cdot \left(E\left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right|^{\frac{2(5+\delta)}{6+\delta}} \right)^{\frac{3+\delta}{2(4+\delta)}} \right).
\end{aligned}$$

Due to the existence of finite $(4+\delta)$-th absolute moments, we can bound term II further by the use of part (ii) of Lemma 2.3. Thus, we have

$$\mathrm{II} \le C \left| \frac{h+r}{T} - \frac{h}{T} \right|^{\frac{3+\delta}{2(4+\delta)}} \le C_1 \, D_T^{\frac{3+\delta}{2(4+\delta)}}$$

because of $|r| \le T D_T$. As explained above, we also get $\mathrm{III} \le C_2 \, D_T^{\frac{3+\delta}{2(4+\delta)}}$ and hence

$$f\left(\underline{s}, \widetilde{\underline{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \widetilde{\underline{X}}_{k+l}\left(\frac{k+l}{T}\right)\right)$$
$$= f\left(\underline{s}, \widetilde{\underline{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \widetilde{\underline{X}}_{k+l}\left(\frac{k}{T}\right)\right) + \mathcal{O}_P\left(D_T^{\frac{3+\delta}{2(4+\delta)}}\right).$$

This ends the proof.

$\square$

Equipped with these additional lemmata, we return to the results of Section 3.3. More precisely, we conduct the proof of Lemma 3.6 up next.

Proof of Lemma 3.6. We start by rewriting the left-hand side of (3.4). Because both t_1 and t_2 do not belong to EP, we obtain

$$\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\Bigg(\Bigg(f\left(\underline{s}, \widetilde{\underline{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \widetilde{\underline{X}}_{t_1+l}(u)\right)\Bigg)$$
$$\cdot\Bigg(f\left(\underline{s}, \widetilde{\underline{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \widetilde{\underline{X}}_{t_2+l}(u)\right)\Bigg)\Bigg)$$
$$= \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\Bigg(\Bigg(f\left(\underline{s}, \widetilde{\underline{X}}_{t_1+r}(u)\right) - Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_1+r}(u)\right) + Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_1+r}(u)\right)$$
$$- \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\left(f\left(\underline{s}, \widetilde{\underline{X}}_{t_1+l}(u)\right) - Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_1+l}(u)\right) + Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_1+l}(u)\right)\right)\Bigg)$$
$$\cdot\Bigg(f\left(\underline{s}, \widetilde{\underline{X}}_{t_2+r}(u)\right) - Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_2+r}(u)\right) + Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_2+r}(u)\right)$$
$$- \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\left(f\left(\underline{s}, \widetilde{\underline{X}}_{t_2+l}(u)\right) - Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_2+l}(u)\right) + Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_2+l}(u)\right)\right)\Bigg)\Bigg)$$
$$= \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\Bigg(\Bigg(\Bigg(\bar{f}\left(\underline{s}, \widetilde{\underline{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \widetilde{\underline{X}}_{t_1+l}(u)\right)\Bigg)$$
$$+ \Bigg(Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_1+l}(u)\right)\Bigg)\Bigg)$$
$$\cdot\Bigg(\Bigg(\bar{f}\left(\underline{s}, \widetilde{\underline{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \widetilde{\underline{X}}_{t_2+l}(u)\right)\Bigg)$$
$$+ \Bigg(Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} Ef\left(\underline{s}, \widetilde{\underline{X}}_{t_2+l}(u)\right)\Bigg)\Bigg)\Bigg)$$

$$= \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left(\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right)\right) \cdot \left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right)\right), \tag{A.85}$$

where $\bar{f}$ signifies the centered version of the function f as defined in (2.15). Next, we want to set $(A.85)$ in relation to the real world covariance by

$$E\left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left(\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right)\right) \cdot \left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right)\right) - \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)^2$$

$$= E\left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) - \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)^2$$

$$\begin{aligned}
&= \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \\
&\quad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right) \\
&\quad + \frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \sum_{m=-TD_T}^{TD_T} \\
&\quad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+m}(u)\right)\right) \\
&\quad - \frac{2}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \\
&\quad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right) \\
&\quad + \frac{2}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right) \\
&\quad \cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right) \\
&\quad - \left(\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)^2 \\
&=: \underline{\mathrm{I}} + \underline{\mathrm{II}} - \underline{\mathrm{III}} + \underline{\mathrm{IV}} - \underline{\mathrm{V}}.
\end{aligned} \tag{A.86}$$

Now we will treat the summands above in two steps. To begin with, Lemma A.1 tells us

$$\underline{\mathrm{II}} \leq \frac{C_{sum,3}}{TD_T}, \quad \underline{\mathrm{III}} \leq \frac{C_{sum,2}}{TD_T} \quad \text{and} \quad \underline{\mathrm{IV}} \leq \frac{C_{sum,1}}{TD_T}.$$

So the difference between $\underline{\mathrm{I}}$ and $\underline{\mathrm{V}}$ of (A.86) is left over. We set $v := t_1 - t_2$, and stationarity helps us to get

$$\begin{aligned}
&|\underline{\mathrm{I}} - \underline{\mathrm{V}}| \\
&= \Bigg| \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right) \\
&\qquad\qquad - \left(\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)^2 \Bigg| \\
&= \Bigg| \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right) \\
&\qquad - E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right)\right) E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right) \Bigg| \\
&= \Bigg| \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \\
&\qquad \mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right) \Bigg| \\
&\leq \frac{2}{(2TD_T+1)^2} \sum_{t=1}^{2TD_T} (2TD_T + 1 - t) \\
&\qquad \cdot \left| \mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t+v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right) \right| \\
&\qquad + \frac{1}{(2TD_T+1)^2} \left| \mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right) \right| \\
&\leq \frac{2}{(2TD_T+1)^2} \sum_{t=1}^{2TD_T} (2TD_T + 1 - t) \\
&\qquad \cdot \left| \mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t+v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right) \right| + \frac{C_1}{(TD_T)^2}.
\end{aligned} \tag{A.87}$$

To deal with the remaining weighted sum of (A.87), the next case-to-case analysis is needed. We start by assuming $v > 0$. Then, we have

$$
\begin{aligned}
&\frac{2}{(2TD_T+1)^2}\sum_{t=1}^{2TD_T}(2TD_T+1-t)\\
&\qquad\qquad\cdot\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\
&\leq\frac{2}{2TD_T+1}\sum_{t=1}^{v-1}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\
&\quad+\frac{2}{2TD_T+1}\sum_{t=v+1}^{2TD_T}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\
&\qquad+\frac{2}{2TD_T+1}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{2v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\
&=:\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
\tag{A.88}
$$

For the last term, it holds $\mathrm{III}\leq\frac{C_2}{TD_T}$. Regarding term II, we have $t>v>0$ and hence $t+v>t>v>0$. Therefore, we set the truncation parameter in (2.13) as $M:=\left\lceil\frac{t-v}{2}\right\rceil$ and get using part (i) of Lemma 3.4

$$
\mathrm{II}\leq\frac{C}{2TD_T+1}\sum_{t=v+1}^{2TD_T}\frac{1}{(t-v)^{1+\tilde{\delta}}}\leq\frac{C}{2TD_T+1}\sum_{t=1}^{2TD_T-v}\frac{1}{t^{1+\tilde{\delta}}}\leq\frac{C_3}{TD_T}.
$$

For term I of equation (A.88), we have $v>t>0$ and thus $t+v>v>t>0$. Therefore, we need two different truncation parameters, namely $M_1:=\lceil t/2\rceil$ and $M_2:=\left\lceil\frac{v-t}{2}\right\rceil$. With those we get by following the second part of Lemma 3.4

$$
\begin{aligned}
\mathrm{I}&=\frac{2}{2TD_T+1}\sum_{t=1}^{v/2}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\
&\qquad+\frac{2}{2TD_T+1}\sum_{t=v/2+1}^{v-1}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\left.\left.\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\\
&\leq\frac{C}{2TD_T+1}\sum_{t=1}^{v/2}\frac{1}{t^{1+\tilde{\delta}}}+\frac{C}{2TD_T+1}\sum_{t=v/2+1}^{v-1}\frac{1}{(v-t)^{1+\tilde{\delta}}}\\
&\leq\frac{C_4}{TD_T}.
\end{aligned}
$$

At this point, the first case is finished. The next would be $v < 0$, but this case can be treated analogously to the first one. Therefore, we move directly on to the last case, which is $v = 0$. Then, the weighted sum in (A.87) equals

$$\frac{2}{(2TD_T+1)^2}\sum_{t=1}^{2TD_T}(2TD_T+1-t) \\ \cdot\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_0(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_0(u)\right)\right)\right|.$$

This gives us $M := \lceil t/2 \rceil$ as truncation parameter, and, once again, we can follow the argumentation of the first case to get the very same result. Summing up, we get $\underline{\mathrm{I}} - \underline{\mathrm{V}} \leq \frac{C}{TD_T}$ and with this

$$\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left(\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u)\right)\right)\right. \\ \left.\cdot\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u)\right)\right)\right) \\ = \operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\right)+\mathcal{O}_P\left(\frac{1}{TD_T}\right),$$

which closes the proof. □

Before we arrive at the next theorem, one lemma is left to prove, namely Lemma 3.7. This happens in the following lines:

Proof of Lemma 3.7. (i) Since t can either be and endpoint or not, there are three different cases to look at, namely:

a) $t \notin EP$,

b) $t \in EP_1$ or

c) $t \in EP_2$.

Starting with the first case, we have

$$E\left|E^\star f\left(\underline{s},\underline{X}^\star_{t,T}\right)\right| \leq \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}E\left|f\left(\underline{s},\underline{X}_{t+r,T}\right)\right| \leq C$$

leading to

$$E^\star f\left(\underline{s},\underline{X}^\star_{t,T}\right) = \mathcal{O}_P(1).$$

Moving on to the endpoint cases, we notice the similar behavior of both due to the symmetrical definition of the endpoint groups. Thus, we only examine case c) further. Afterwards, the results can easily be transferred to the remaining case. Hence, let t be part of EP_2. Then, it holds

$$E^{\star} f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right) = \frac{1}{2TD_T+1}\left(\sum_{r=-TD_T}^{T-t} f\left(\underline{s}, \underline{X}_{t+r,T}\right) + \sum_{r=T-t+1}^{TD_T} f\left(\underline{s}, \underline{X}_{t-r,T}\right)\right).$$

Consequently, we get

$$\begin{aligned} & E\left|E^{\star} f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right)\right| \\ & \leq \frac{1}{2TD_T+1}\left(\sum_{r=-TD_T}^{T-t} E\left|f\left(\underline{s}, \underline{X}_{t+r,T}\right)\right| + \sum_{r=T-t+1}^{TD_T} E\left|f\left(\underline{s}, \underline{X}_{t-r,T}\right)\right|\right) \\ & \leq C. \end{aligned}$$

This finishes case c) and therewith, as explained above, case b) as well.

(ii) As both indices t_1 and t_2 appertain to the same bootstrap block, we can distinguish between five different cases:

a) $t_1, t_2 \notin EP$,

b) $t_1, t_2 \in EP_1$,

c) $t_1, t_2 \in EP_2$,

d) $t_1 \in EP_1$, $t_2 \notin EP$ or

e) $t_1 \in EP_2$, $t_2 \notin EP$.

Because the relation between t_1 and t_2 is not specified, we can change their roles if needed, for example, if we have $t_2 \in EP_1$ without t_2 being an endpoint.

In the beginning, consider case a). We rewrite the covariance as follows:

$$\begin{aligned} & \operatorname{Cov}^{\star}\left(f\left(\underline{s}, \underline{X}_{t_1,T}^{\star}\right), f\left(\underline{s}, \underline{X}_{t_2,T}^{\star}\right)\right) \\ & = \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left(\left(f\left(\underline{s}, \underline{X}_{t_1+r,T}^{\star}\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{t_1+l,T}^{\star}\right)\right)\right. \\ & \qquad \left.\cdot\left(f\left(\underline{s}, \underline{X}_{t_2+r,T}^{\star}\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{t_2+l,T}^{\star}\right)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right) \\
&\qquad\qquad - \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right) \\
&\quad - \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_1+l,T}\right) \\
&\quad + \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right) \\
&= \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right) \\
&\qquad\qquad - \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&E\left|\mathrm{Cov}^{\star}\left(f\left(\underline{s}, \underline{X}^{\star}_{t_1,T}\right), f\left(\underline{s}, \underline{X}^{\star}_{t_2,T}\right)\right)\right| \\
&\leq \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} E\left|f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right)\right| \\
&\qquad\qquad + \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} E\left|f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right)\right| \\
&\leq C_1
\end{aligned}
$$

since f has finite absolute moments of order $4+\delta$, which implies the existence of finite second moments. Thus, it holds

$$
\mathrm{Cov}^{\star}\left(f\left(\underline{s}, \underline{X}^{\star}_{t_1,T}\right), f\left(\underline{s}, \underline{X}^{\star}_{t_2,T}\right)\right) = \mathcal{O}_P(1).
$$

Now we move on to the endpoint cases. Thereby, we focus on the case where $t_1, t_2 \in EP_2$, that is c), since the result for $t_1, t_2 \in EP_1$ can be derived similarly. W.l.o.g. we assume $t_1 \geq t_2$, which implies $T - t_2 \geq T - t_1$. Hence, we get

$$
\begin{aligned}
&E\left|\operatorname{Cov}^\star\left(f\left(\underline{s},\underline{X}^\star_{t_1,T}\right),f\left(\underline{s},\underline{X}^\star_{t_2,T}\right)\right)\right| \\
&\leq \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{T-t_1} \\
&E\Bigg|\left(f\left(\underline{s},\underline{X}^\star_{t_1+r,T}\right)-\frac{1}{2TD_T+1}\left(\sum_{l=-TD_T}^{T-t_1}f\left(\underline{s},\underline{X}^\star_{t_1+l,T}\right)+\sum_{l=T-t_1+1}^{TD_T}f\left(\underline{s},\underline{X}^\star_{t_1-l,T}\right)\right)\right) \\
&\quad\cdot\left(f\left(\underline{s},\underline{X}^\star_{t_2+r,T}\right)-\frac{1}{2TD_T+1}\left(\sum_{l=-TD_T}^{T-t_2}f\left(\underline{s},\underline{X}^\star_{t_2+l,T}\right)+\sum_{l=T-t_2+1}^{TD_T}f\left(\underline{s},\underline{X}^\star_{t_2-l,T}\right)\right)\right)\Bigg| \\
&\qquad\qquad\qquad + \frac{1}{2TD_T+1}\sum_{r=T-t_1+1}^{T-t_2} \\
&E\Bigg|\left(f\left(\underline{s},\underline{X}^\star_{t_1-r,T}\right)-\frac{1}{2TD_T+1}\left(\sum_{l=-TD_T}^{T-t_1}f\left(\underline{s},\underline{X}^\star_{t_1+l,T}\right)+\sum_{l=T-t_1+1}^{TD_T}f\left(\underline{s},\underline{X}^\star_{t_1-l,T}\right)\right)\right) \\
&\quad\cdot\left(f\left(\underline{s},\underline{X}^\star_{t_2+r,T}\right)-\frac{1}{2TD_T+1}\left(\sum_{l=-TD_T}^{T-t_2}f\left(\underline{s},\underline{X}^\star_{t_2+l,T}\right)+\sum_{l=T-t_2+1}^{TD_T}f\left(\underline{s},\underline{X}^\star_{t_2-l,T}\right)\right)\right)\Bigg| \\
&\qquad\qquad\qquad + \frac{1}{2TD_T+1}\sum_{r=T-t_2+1}^{TD_T} \\
&E\Bigg|\left(f\left(\underline{s},\underline{X}^\star_{t_1-r,T}\right)-\frac{1}{2TD_T+1}\left(\sum_{l=-TD_T}^{T-t_1}f\left(\underline{s},\underline{X}^\star_{t_1+l,T}\right)+\sum_{l=T-t_1+1}^{TD_T}f\left(\underline{s},\underline{X}^\star_{t_1-l,T}\right)\right)\right) \\
&\quad\cdot\left(f\left(\underline{s},\underline{X}^\star_{t_2-r,T}\right)-\frac{1}{2TD_T+1}\left(\sum_{l=-TD_T}^{T-t_2}f\left(\underline{s},\underline{X}^\star_{t_2+l,T}\right)+\sum_{l=T-t_2+1}^{TD_T}f\left(\underline{s},\underline{X}^\star_{t_2-l,T}\right)\right)\right)\Bigg| \\
&=: \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
$$

As the way I, II and III are constructed is the same, we only look at term I and get the same result for terms II and III. It holds

$$
\begin{aligned}
\mathrm{I} \leq{} & \frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{T-t_1} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right)\right| \\
& \qquad + \frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{T-t_1} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right)\right. \\
& \qquad \left. \cdot \frac{1}{2\,TD_T+1} \left(\sum_{l=-TD_T}^{T-t_2} f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right) + \sum_{l=T-t_2+1}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_2-l,T}\right) \right)\right| \\
& \qquad + \frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{T-t_1} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right)\right. \\
& \qquad \left. \cdot \frac{1}{2\,TD_T+1} \left(\sum_{l=-TD_T}^{T-t_1} f\left(\underline{s}, \underline{X}^{\star}_{t_1+l,T}\right) + \sum_{l=T-t_1+1}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_1-l,T}\right) \right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} E\left| \left(\sum_{r=-TD_T}^{T-t_1} f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) + \sum_{r=T-t_1+1}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_1-r,T}\right) \right)\right. \\
& \qquad \left. \cdot \left(\sum_{r=-TD_T}^{T-t_2} f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right) + \sum_{r=T-t_2+1}^{TD_T} f\left(\underline{s}, \underline{X}^{\star}_{t_2-r,T}\right) \right)\right| \\
\leq{} & C_2 + \frac{1}{(2\,TD_T+1)^2} \sum_{r=-TD_T}^{T-t_1} \sum_{l=-TD_T}^{T-t_2} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} \sum_{r=-TD_T}^{T-t_1} \sum_{l=T-t_2+1}^{TD_T} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2-l,T}\right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} \sum_{r=-TD_T}^{T-t_1} \sum_{l=-TD_T}^{T-t_1} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_1+l,T}\right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} \sum_{r=-TD_T}^{T-t_1} \sum_{l=T-t_1+1}^{TD_T} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_2+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_1-l,T}\right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} \sum_{r=-TD_T}^{T-t_1} \sum_{l=-TD_T}^{T-t_2} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} \sum_{r=-TD_T}^{T-t_1} \sum_{l=T-t_2+1}^{TD_T} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2-l,T}\right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} \sum_{r=T-t_1+1}^{TD_T} \sum_{l=-TD_T}^{T-t_2} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1-r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2+l,T}\right)\right| \\
& \qquad + \frac{1}{(2\,TD_T+1)^2} \sum_{r=T-t_1+1}^{TD_T} \sum_{l=T-t_2+1}^{TD_T} E\left| f\left(\underline{s}, \underline{X}^{\star}_{t_1-r,T}\right) f\left(\underline{s}, \underline{X}^{\star}_{t_2-l,T}\right)\right| \\
\leq{} & C_3.
\end{aligned}
$$

Therefore, we have

$$\mathrm{Cov}^{\star}\left(f\left(\underline{s}, \underline{X}^{\star}_{t_1,T}\right), f\left(\underline{s}, \underline{X}^{\star}_{t_2,T}\right)\right) = \mathcal{O}_P(1).$$

The remaining cases can be treated using a combination of the calculations done both in the endpoint and non-endpoint cases. This completes the proof of the second part and thus the proof as a whole.

□

As already insinuated, we have reached the proof of the next theorem, whose index segmentation will be pioneering for the upcoming proofs.

Proof of Theorem 3.8. Before we start with the proof of the result in question, we have a look at the ratio between the number of indices in the endpoint groups EP_1 and EP_2 and the number of indices belonging to one bootstrap block. The question we like to answer is whether all indices of one endpoint group can theoretically be found in a single bootstrap block of length L_T. Using Assumption 6, we can deduce $L_T < d_T^{-\frac{\delta}{2(1+\delta)}}$ for T large enough. Consulting once again said assumption, we know $TD_T \geq d_T^{-\frac{\delta}{2+\delta}}$. In order to find the answer, we compare both exponents. Assuming a negative reply, we aim to validate the following inequality:

$$\frac{\delta}{2+\delta} - \frac{\delta}{2(1+\delta)} > 0.$$

This is equivalent to $2 + 2\delta - 2 - \delta = \delta > 0$, which is fulfilled for all $\delta \in (0, 1)$. Hence, we can state that for T sufficiently large it is not possible to find all indices appurtenant to either EP_1 or EP_2 in one single bootstrap block with our chosen length. Throughout this proof, we will assume that T is large enough to ensure the validity of the just made conclusion. In this context, we move on to the main part of the proof and start by dividing the sum in question into three sums, of which two only contain endpoints. Thus, we have

$$\begin{aligned}
&\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \\
&\quad = \sum_{t=1}^{TD_T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) + \sum_{t=TD_T+1}^{T-TD_T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) + \sum_{t=T-TD_T+1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right). \qquad \text{(A.89)}
\end{aligned}$$

Since the indices of each endpoint group do not form a single bootstrap block, we can conclude that the middle sum from above do not consist of whole blocks either. This leads to a de novo partitioning of the middle sum of (A.89) to separate the part containing whole blocks. Therefore, we get

$$
\begin{aligned}
&\sum_{t=TD_T+1}^{T-TD_T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right) \\
&= \sum_{t=TD_T+1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right) + \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right) \\
&\qquad + \sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T-TD_T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right). \qquad \text{(A.90)}
\end{aligned}
$$

Next, we combine the separated sums from (A.89) and (A.90) which are not composed of whole bootstrap blocks and obtain

$$
\begin{aligned}
&\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right) \\
&= \sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right) + \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right) \\
&\qquad + \sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right).
\end{aligned}
$$

Due to the independence of the bootstrap blocks, this segmentation can be transferred to the bootstrap variance, that is

$$
\begin{aligned}
&\operatorname{Var}^{\star}\left(\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right)\right) \\
&= \operatorname{Var}^{\star}\left(\sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right)\right) + \operatorname{Var}^{\star}\left(\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right)\right) \\
&\qquad + \operatorname{Var}^{\star}\left(\sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}^{\star}\right)\right) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad \text{(A.91)}
\end{aligned}
$$

Now we examine the newly defined variances apartly. Since terms I and III from (A.91) are of the same building type, we focus on the first. We use Lemma part (i) of 3.7 and the rates from Assumption 6 to establish a bound for this bootstrap variance, that is

$$
\begin{aligned}
\mathrm{I} &= \sum_{t_1,t_2=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t_1,T}\, w_{t_2,T} \operatorname{Cov}^{\star}\left(f\left(\underline{s}, \underline{X}_{t_1,T}^{\star}\right), f\left(\underline{s}, \underline{X}_{t_2,T}^{\star}\right)\right) \\
&= \mathcal{O}_P\left(\left(TD_T\right)^2 d_T\right)
\end{aligned}
$$

$$= \mathcal{O}_P\left(d_T^{\frac{\delta}{2+\delta}}\right), \tag{A.92}$$

which tends to 0 as $T \to \infty$. As explained above, the same holds true for term III. Thus, we only need to set our focus on the second bootstrap variance term of (A.91). Once again, we make use of the independence of the bootstrap blocks and get by transforming the variance of the sum into sums of covariances

$$\begin{aligned}
\mathrm{II} = & \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}\ \mathrm{Cov}^\star\left(f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right), f\left(\underline{s}, \underline{X}^\star_{tL_T+l,T}\right)\right) \\
= & \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \\
& \cdot \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{tL_T+j+r,T}\right) f\left(\underline{s}, \underline{X}_{tL_T+l+r,T}\right)\right. \\
& \left. - \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{tL_T+j+r,T}\right) \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{tL_T+l+k,T}\right)\right).
\end{aligned} \tag{A.93}$$

We aim for transforming the bootstrap covariance into the real world covariance with negligible error. To do so, the first step is to change the process $\left(\underline{X}_{t,T}\right)$ to the companion process. Lemma A.2 helps us with it, and (A.93) becomes

$$\begin{aligned}
& \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \\
& \cdot \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\right)\right. \\
& \qquad + \mathcal{O}_P\left(T^{-\frac{3+\delta}{2(4+\delta)}}\right) \\
& \quad - \frac{1}{(2TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right. \\
& \qquad \left.\left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right)\right) + \mathcal{O}_P\left(T^{-\frac{3+\delta}{2(4+\delta)}}\right)\right)
\end{aligned}$$

$$\leq \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}$$
$$\cdot \Bigg(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\right)$$
$$- \frac{1}{(2TD_T+1)^2} \Bigg(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)$$
$$\cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right)\Bigg)\Bigg) + \mathcal{O}_P\left(\frac{L_T}{T^{\frac{3+\delta}{2(4+\delta)}}}\right). \tag{A.94}$$

Next, we show that the denominator of $\mathcal{O}_P\left(\frac{L_T}{T^{\frac{3+\delta}{2(4+\delta)}}}\right)$ grows faster than the numerator. Then, it holds $\mathcal{O}_P\left(\frac{L_T}{T^{\frac{3+\delta}{2(4+\delta)}}}\right) \subseteq o_P(1)$, and we concentrate on the first summand of (A.94). Since we assumed T to be large enough to satisfy $L_T < d_T^{-\frac{\delta}{2(1+\delta)}}$, we have $L_T < T^{\frac{\delta}{2(1+\delta)}}$ as well. This allows for the mere comparison of the exponents. If we can verify

$$\frac{\delta}{2(1+\delta)} < \frac{3+\delta}{2(4+\delta)}, \tag{A.95}$$

the inclusion of the $\mathcal{O}_P$-term in question in the class $o_P(1)$ follows immediately. Equation (A.95) is equivalent to $4\delta + \delta^2 < 3 + 4\delta + \delta^2$. Clearly, this holds true, and we can neglect $\mathcal{O}_P\left(\frac{L_T}{T^{\frac{3+\delta}{2(4+\delta)}}}\right)$ for T tending to ∞. Thus, we move on with the remaining summand of (A.94). To turn it into a sum of real world covariances, we have change the argument of $\left(\underline{\widetilde{X}}_t(u)\right)$, so that it loses the dependence of the inner summation index. But before doing that, we rewrite the sums to be able to summate over the covariance lag. This lag will be denoted by h. Therefore, the first summand of (A.94) becomes

$$\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T, L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \Bigg(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right)$$
$$\cdot f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)$$
$$- \frac{1}{(2TD_T+1)^2} \Bigg(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right)$$
$$\cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j+k}{T}\right)\right)\Bigg)\Bigg). \tag{A.96}$$

At this point, we use the second part of Lemma A.3 to perform the aforementioned change of the argument and get

$$\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T, L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\right) \right.$$
$$- \frac{1}{(2TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right) \right.$$
$$\left.\left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\right)\right)\right)$$
$$+ \mathcal{O}_P\left(L_T\, D_T^{\frac{3+\delta}{2(4+\delta)}}\right) \tag{A.97}$$

instead of (A.96). Now the arguments do not longer depend on the summation indices, which allows for the use of Lemma 3.6 to transform the bootstrap covariance into the real world one. Before moving on, we have a closer look at $\mathcal{O}_P\left(L_T\, D_T^{\frac{3+\delta}{2(4+\delta)}}\right)$. Because of Assumption 6, it holds

$$\begin{aligned} L_T\, D_T^{\frac{3+\delta}{2(4+\delta)}} &\leq d_T^{-\frac{\delta}{2(1+\delta)}} \left(T d_T^{\frac{1}{2+\delta}}\right)^{-\frac{3+\delta}{2(4+\delta)}} \\ &\leq d_T^{-\frac{\delta}{2(1+\delta)}} \left(d_T^{-\frac{1+\delta}{2+\delta}}\right)^{-\frac{3+\delta}{2(4+\delta)}} \\ &= d_T^{-\frac{\delta}{2(1+\delta)}+\frac{1+\delta}{2+\delta}\cdot\frac{3+\delta}{2(4+\delta)}}. \end{aligned} \tag{A.98}$$

Therefore, to show $\mathcal{O}_P\left(L_T\, D_T^{\frac{3+\delta}{2(4+\delta)}}\right)$ is part of the class $o_P(1)$, we need to ensure that the exponent in equation (A.98) is positive. This is equivalent to the following inequality

$$-8\delta - 6\delta^2 - \delta^3 + \left(1 + 2\delta + \delta^2\right) \cdot (3+\delta).$$

Expansion of the product and pooling suitable terms results in $3 - \delta - \delta^2 > 0$, which is satisfied by every $\delta \in (0,1)$. Hence, we focus again on the first summand of (A.97) and obtain

$$\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \Bigg(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\right)$$
$$- \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right)\right.$$
$$\left.\cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\right)\right)\Bigg)$$
$$= \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \left(\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h}\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j}\left(\frac{tL_T+j}{T}\right)\right)\right) + \mathcal{O}_P\left(\frac{1}{TD_T}\right)\right)$$
$$= \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot\, \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{h}\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{0}\left(\frac{tL_T+j}{T}\right)\right)\right) + \mathcal{O}_P\left(\frac{L_T}{TD_T}\right) \tag{A.99}$$

to work with. As before, we show $\mathcal{O}_P\left(\frac{L_T}{TD_T}\right)$ belongs to the class $o_p(1)$. In order to do so, we will bound the numerator from above, whereas the denominator will be bounded from below. If the latter bound increases faster than the former, we obtain the preferred result. Following our assumption from the beginning, it holds $L_T < d_T^{-\frac{\delta}{2(1+\delta)}}$. As opposed to this, we have $TD_T \geq d_T^{-\frac{\delta}{2+\delta}}$ by Assumption 6. Since both bounds have the same base, we transfer the comparison to the exponents anew. Because of

$$\frac{\delta}{2(1+\delta)} < \frac{\delta}{2+\delta}$$

being equivalent to $\delta^2 < 2$, the denominator grows faster than the numerator given any $\delta \in (0,1)$. Thus, we focus on the sums of the weighted real world covariances. Comparably to the proof of Lemma 2.13, we would like to rewrite the inner sum in order to eliminate the minimum and maximum determining the index bounds. Therefore, we show that the difference between the first summand of (A.99) and

$$\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=1}^{L_T} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right) \quad \text{(A.100)}$$

is negligible as T tends to infinity. To make sure (A.100) is well-defined, we use the same notation modification as in the proof of Lemma 2.13, that is setting

$$\underline{\widetilde{X}}_t(z) = \begin{cases} \underline{\widetilde{X}}_t(1), & z > 1, \\ \underline{\widetilde{X}}_t(0), & z < 0, \end{cases}$$

for all $t \in \mathbb{Z}$ and

$$w_{t,T} = \begin{cases} w_{T,T}, & t > T, \\ w_{1,T}, & t < 1. \end{cases}$$

Now we have a look at the difference in question and get

$$\Bigg| \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)$$
$$- \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=1}^{L_T} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)\Bigg|$$
$$\leq \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=1}^{L_T-1} \sum_{j=L_T-h+1}^{L_T} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \left|\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)\right|$$
$$+ \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{-1} \sum_{j=1}^{-h} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \left|\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)\right|$$
$$\leq \frac{C_1}{L_T} \sum_{h=1}^{L_T-1} h \sum_{|k|\geq \lceil h/2\rceil} \frac{B}{l(k)}$$
$$\leq \frac{C_1}{L_T} \sum_{h=1}^{L_T-1} \sum_{|k|\geq \lceil h/2\rceil} k \frac{B}{l(k)}$$

$$
\begin{aligned}
&\leq \frac{C_1}{L_T} \sum_{k \in \mathbb{Z}} k^2 \frac{B}{l(k)} \\
&= \mathcal{O}\left(L_T^{-1}\right)
\end{aligned}
$$

because of $\frac{\left(1+\tilde{\delta}\right)(3+\delta)}{\delta} > 2$ and Assumption 5. Since L_T tends to infinity as $T \to \infty$, $\mathcal{O}\left(L_T^{-1}\right)$ is contained in $o_P(1)$. Thus, we can proceed with (A.100). Letting the first two sums change place gives

$$
\begin{aligned}
&\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} w_{tL_T+j+h,T}\, w_{tL_T+j,T} \\
&\qquad \cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right). \qquad \text{(A.101)}
\end{aligned}
$$

The next step will be to incorporate the sum over j back into the sum over t. Thereby, we obtain

$$
\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil+1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \qquad \text{(A.102)}
$$

instead of (A.101). The last step is to change the argument of $\underline{\widetilde{X}}_h(\cdot)$ in (A.102) to match the one of $\underline{\widetilde{X}}_0(\cdot)$, that is to eliminate h in the numerator. By following exactly the lines of the proof of Lemma 2.13, we can bound (A.102) by

$$
\begin{aligned}
&\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil+1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \\
&\qquad \cdot \left(\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) + \mathcal{O}\left(\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}}\right)\right) \\
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil+1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \\
&\qquad + \mathcal{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right). \qquad \text{(A.103)}
\end{aligned}
$$

Since the last term of (A.103) tends to 0 as T tends to infinity the way L_T is defined, we focus on the first one. It remains to show

$$\lim_{T\to\infty} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) = \sum_{h=-\infty}^{\infty} V_h\left(\underline{s}, \underline{s}\right) \quad \text{(A.104)}$$

with $V_h\left(\underline{s}_1, \underline{s}_2\right)$ defined in Lemma 2.13. We continue by incorporating the limit into the outer sum of the right-hand side of (A.104). As in the proof of Lemma (2.13), we need to ensure that an upper bound capable to be totaled for the inner sum of the right-hand side from above exists. We start by bounding the covariance. For this purpose, remember the case-by-case analysis made in the proof of Lemma (2.13) concerning the values h can adopt. First, we look at $h \neq 0$. In this case, Lemma 2.11 tells us

$$\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \leq \frac{C_{Cov}}{|h|^{1+\tilde{\delta}}}.$$

On the other hand, we get

$$\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) = \mathrm{Var}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \leq C_2$$

for $h = 0$ using the Cauchy-Schwarz inequality. Consequently, we have

$$\begin{aligned}
&\left|\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\left(\frac{C_{Cov}}{|h|^{1+\tilde{\delta}}}\mathbb{1}_{\{h\neq 0\}} + C_2\mathbb{1}_{\{h=0\}}\right) \\
&\leq C_2\left(|h|^{-\left(1+\tilde{\delta}\right)}\mathbb{1}_{\{h\neq 0\}} + \mathbb{1}_{\{h=0\}}\right)
\end{aligned}$$

as an upper bound for the inner sum of (A.104), which is capable to be totaled because of

$$\sum_{h=-\infty}^{\infty} C_2\left(|h|^{-\left(1+\tilde{\delta}\right)}\mathbb{1}_{\{h\neq 0\}} + \mathbb{1}_{\{h=0\}}\right) = C_2\left(1 + 2\sum_{h=1}^{\infty} h^{-\left(1+\tilde{\delta}\right)}\right) < \infty.$$

Thus, Lebesgue's theorem allows us to work with

$$\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \quad \text{(A.105)}$$

from now on. The next step will be to adjust the summation bounds. We would like to go on with

$$\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \tag{A.106}$$

in lieu of (A.105). Thus, we examine the difference between the two expressions, that is

$$\Bigg|\sum_{t=1}^{T} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \\ - \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\Bigg|.$$

This difference can be bounded by

$$\begin{aligned}
&\sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t+h,T}\, w_{t,T}\left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\right| \\
&\qquad + \sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} w_{t+h,T}\, w_{t,T}\left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq \left(TD_T + L_T\right)^2 d_T\, C_3 \\
&= \mathcal{O}\left(d_T^{\frac{\delta}{2+\delta}}\right)
\end{aligned}$$

using the calculations made in equation (A.92). This justifies the use of (A.106) and finishes the proof as this expression equals $\sum_{h=-\infty}^{\infty} V_h\left(\underline{s}, \underline{s}\right)$. □

To finish this subsection, the proof of the eponymous theorem, the bootstrap CLT, is next in line. It is inspired by the one Dowla et al. (2013) proposed but enhanced to meet the demands imposed by our assumptions.

Proof of Theorem 3.9. Just as in the proof of Theorem 3.8, the original sum can be split up in one sum containing all indices belonging to whole bootstrap blocks without including endpoints and two with the remaining indices in the way that we have

$$\begin{aligned}
&\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \\
&= \sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) + \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \\
&\qquad + \sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right)
\end{aligned}$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \tag{A.107}$$

Again, comparable to the aforementioned proof, we assume the validity of $L_T < d_T^{-\frac{\delta}{2(1+\delta)}}$. Then, can bound the first term of (A.107) via

$$\begin{aligned}
\mathrm{I} &\leq \left| \sum_{t=1}^{L_T \lceil (TD_T+1)/L_T \rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}^\star_{t,T}\right) \right| \\
&= \mathcal{O}_P\left(\left(TD_T + L_T\right) d_T^{1/2} \right) \\
&= \mathcal{O}_P\left(TD_T\, d_T^{1/2} \right) \\
&= \mathcal{O}_P\left(d_T^{\frac{\delta}{2(2+\delta)}} \right)
\end{aligned}$$

due to Assumption 6 and the first part of Lemma 3.7. Because of the similar structure, we obtain $\mathrm{III} = \mathcal{O}_P\left(d_T^{\frac{\delta}{2(2+\delta)}} \right)$ in the same manner. As these rates belong to the class $o_P(1)$, we can turn our attention to the remaining sum of (A.107) containing the whole blocks. This sum can be rewritten to make the single blocks visible, to wit

$$\mathrm{II} = \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{j=1}^{L_T} w_{tL_T+j,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) =: \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \xi^\star_{t,T}.$$

Since the bootstrap blocks are independent, the newly defined $\left(\xi^\star_{t,T}\right)$ are independent as well. Therefore, we aim for applying the classical central limit theorem using Lyapunov's condition. Theorem 3.8 gives us

$$P - \lim_{T\to\infty} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \mathrm{Var}^\star\left(\xi^\star_{t,T}\right) = \sigma^2\left(\underline{s}, \underline{s}\right). \tag{A.108}$$

The next step is to verify that $\left(\xi^\star_{t,T}\right)$ has finite $(2+\delta)$-th absolute moments. We start by rewriting the moment in question in order to expand the abbreviated form used for the function f and obtain

$$\begin{aligned}
&E^\star \left|\xi^\star_{t,T}\right|^{2+\delta} \\
&= E^\star \left| \sum_{j=1}^{L_T} w_{tL_T+j,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right|^{2+\delta} \\
&= \left\| \sum_{j=1}^{L_T} w_{tL_T+j,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right\|_{2+\delta,\star}^{2+\delta} \\
&\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) - E^\star\left(f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right)\right) \right\|_{2+\delta,\star} \right)^{2+\delta}
\end{aligned}$$

$$\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(\left\| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right\|_{2+\delta,\star} + \left| E^\star \left(f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right)\right) \right| \right) \right)^{2+\delta}$$
$$=: \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(\breve{\mathrm{I}} + \breve{\mathrm{II}}\right) \right)^{2+\delta} .$$

Now we transfer the bootstrap moment into the real world by

$$\begin{aligned} E\left(E^\star \left|\xi^\star_{t,T}\right|^{2+\delta} \right) &\leq E \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(\breve{\mathrm{I}} + \breve{\mathrm{II}}\right) \right)^{2+\delta} \\ &= \left\| \sum_{j=1}^{L_T} w_{tL_T+j,T} \left(\breve{\mathrm{I}} + \breve{\mathrm{II}}\right) \right\|_{2+\delta}^{2+\delta} \\ &\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| \breve{\mathrm{I}} + \breve{\mathrm{II}} \right\|_{2+\delta} \right)^{2+\delta} \\ &\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(\left\|\breve{\mathrm{I}}\right\|_{2+\delta} + \left\|\breve{\mathrm{II}}\right\|_{2+\delta} \right) \right)^{2+\delta} . \end{aligned} \tag{A.109}$$

At this moment we can treat $\left\|\breve{\mathrm{I}}\right\|_{2+\delta}$ and $\left\|\breve{\mathrm{II}}\right\|_{2+\delta}$ independently. For simplicity reasons, we leave the root out of consideration during the first calculations. According to this, we get

$$\begin{aligned} E\breve{\mathrm{I}}^{2+\delta} &= E \left\| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right\|_{2+\delta,\star}^{2+\delta} \\ &= E \left(\left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right|^{2+\delta} \right)^{\frac{1}{2+\delta}} \right)^{2+\delta} \\ &= \frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} E \left| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right|^{2+\delta} \\ &\leq C_1 \end{aligned}$$

and

$$\begin{aligned} E\breve{\mathrm{II}}^{2+\delta} &= E\left(E^\star \left| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right| \right)^{2+\delta} \\ &= \left\| E^\star \left| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right| \right\|_{2+\delta}^{2+\delta} \\ &= \left\| \frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right| \right\|_{2+\delta}^{2+\delta} \end{aligned}$$

$$\leq \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} \left\| f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right) \right\|_{2+\delta} \right)^{2+\delta}$$
$$\leq C_2.$$

Returning to (A.109), we obtain in consequence

$$\begin{aligned} E\left(E^\star \left|\xi^\star_{t,T}\right|^{2+\delta}\right) &\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(\left\| \breve{\mathrm{I}} \right\|_{2+\delta} + \left\| \breve{\mathrm{II}} \right\|_{2+\delta} \right) \right)^{2+\delta} \\ &\leq \left(\sum_{j=1}^{L_T} d_T^{1/2}\, \mathbb{1}_{\left\{w_{tL_T+j,T}>0\right\}} C \right)^{2+\delta} \\ &\leq C\, d_T^{\frac{2+\delta}{2}} \left(\sum_{j=1}^{L_T} \mathbb{1}_{\left\{w_{tL_T+j,T}>0\right\}} \right)^{2+\delta}. \end{aligned} \tag{A.110}$$

At this point, we have made the necessary steps to be able to verify that Lyapunov's condition is fulfilled. Before we do that, note that it holds

$$\begin{aligned} &\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \left(\sum_{j=1}^{L_T} \mathbb{1}_{\left\{w_{tL_T+j,T}>0\right\}} \right)^{2+\delta} \\ &= \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \mathbb{1}_{\left\{w_{tL_T+j,T}>0\right\}} \left(\sum_{j=1}^{L_T} \mathbb{1}_{\left\{w_{tL_T+j,T}>0\right\}} \right)^{1+\delta} \\ &\leq d_T^{-1} L_T^{1+\delta}. \end{aligned} \tag{A.111}$$

Now, combining (A.108) and (A.110) and using (A.111), we obtain

$$\frac{\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} E^\star \left|\xi^\star_{t,T}\right|^{2+\delta}}{\left(\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \mathrm{Var}^\star\left(\xi^\star_{t,T}\right)\right)^{\frac{2+\delta}{2}}} = \frac{d_T^{-1} L_T^{1+\delta} \mathcal{O}_P\left(d_T^{\frac{2+\delta}{2}}\right)}{\left(\sigma^2\left(\underline{s},\underline{s}\right)\right)^{\frac{2+\delta}{2}} + o_P(1)} = \mathcal{O}_P\left(L_T^{1+\delta}\, d_T^{\delta/2}\right).$$

Consulting Assumption 6 anew, $L_T^{1+\delta}\, d_T^{\delta/2}$ tends to 0 as $T \to \infty$. Thereby, Lyapunov's condition is fulfilled. Because of the continuity of the Gaussian distribution function, this implies

$$\sup_{v\in\mathbb{R}} \left| P^\star\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}^\star_{t,T}\right) \leq v \right) - \Phi\left(\frac{v}{\sigma\left(\underline{s},\underline{s}\right)} \right) \right| \xrightarrow{P} 0$$

as T tends to ∞ with Lemma 2.11 of van der Vaart (1998). Thus, the proof is brought to completion. $\square$

A.2.2. Proofs of Section 3.4

In Section 3.4, we made the acquaintance of so-called good sets. The theory explained there will now find its implementation in this subsection. Furthermore, we address the proofs of the results already presented in Section 3.4.

If we have a closer look at the proof of the part of Lemma 2.18 dealing with an unbounded function f, we see that at some point, the establishment of an upper bound was necessary. This is the case in the bootstrap counterpart as well. However, we do not wait until the proof of the tightness itself takes place. Instead, we bring this boundary result forward and combine it with the establishment of our first T-depending sequence of good sets.

Lemma A.4.
Suppose Assumptions 5 and 7 hold true. Then, there exist subsets $(A_T)_{T\in\mathbb{N}}$ of Ω satisfying $P(A_T) \to 1$ as $T \to \infty$ such that it holds

$$\sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\dots,T}} w_{t,T} \left| f\left(\underline{s}, \underline{X}^\star_{t,T}\right)\right| \le d_T^{\frac{2+\delta^2}{2(4+\delta)}}$$

on A_T for the particular T.

Proof. Define

$$A_T := \left\{ \omega \in \Omega \;\middle|\; \sup_{\underline{s}\in[-S,S]^d \;\, t=1,\dots,T} w_{t,T} \left| f\left(\underline{s}, \underline{X}^\star_{t,T}\right)\right| \le d_T^{\frac{2+\delta^2}{2(4+\delta)}} \right\}.$$

Now we verify $P(A_T)$ tends to 1 as $T \to \infty$ by showing that $P\left((A_T)^C\right)$ tends to 0 with $T \to \infty$. First, we insert a self-canceling term in order to make use of the Lipschitz condition (2.11) later on and get

$$\begin{aligned} P\left(A_T^C\right) &= P\left(\sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\dots,T}} w_{t,T} \left| f\left(\underline{s}, \underline{X}^\star_{t,T}\right)\right| > d_T^{\frac{2+\delta^2}{2(4+\delta)}} \right) \\ &\le P\left(\sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\dots,T}} w_{t,T} \left| f\left(\underline{s}, \underline{X}^\star_{t,T}\right) - f\left(\underline{0}, \underline{X}^\star_{t,T}\right)\right| > \frac{d_T^{\frac{2+\delta^2}{2(4+\delta)}}}{2} \right) \\ &\qquad + P\left(\sup_{\substack{\underline{s}\in[-S,S]^d \\ t=1,\dots,T}} w_{t,T} \left| f\left(\underline{0}, \underline{X}^\star_{t,T}\right)\right| > \frac{d_T^{\frac{2+\delta^2}{2(4+\delta)}}}{2} \right) \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{A.112}$$

Next, we benefit from said Lipschitz condition and obtain for the first term in (A.112) with Assumption 3 and the finite moments of order $4+\delta$ the function g has under the use of Markov's inequality the following:

$$
\begin{aligned}
\mathrm{I} &\leq P\left(\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\, g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right)|\underline{s}-\underline{0}| > \frac{d_T^{\frac{2+\delta^2}{2(4+\delta)}}}{2}\right)\\
&\leq 2^{4+\delta}(Sd)^{4+\delta}\sum_{t=1}^{T} w_{t,T}^{4+\delta} E\left(g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right)\right)^{4+\delta} d_T^{-\frac{2+\delta^2}{2}}\\
&\leq C_1\, d_T^{-1} d_T^{\frac{2+\delta-\delta^2}{2}}\\
&= C_1\, d_T^{\frac{\delta-\delta^2}{2}},
\end{aligned}
$$

which tends to 0 as $T\to\infty$. For the second summand in equation (A.112), II, we have again the use of Assumption 3 and the finite $(4+\delta)$-th moments of g

$$
\mathrm{II} \leq 2^{4+\delta}\sum_{t=1}^{T} w_{t,T}^{4+\delta} E\left|f\left(\underline{0}, \underline{X}^\star_{t,T}\right)\right|^{4+\delta} d_T^{-\frac{2+\delta^2}{2}} \leq C_2\, d_T^{\frac{\delta-\delta^2}{2}},
$$

which tends to 0 as $T\to\infty$ as well. Combining both upper bounds we get

$$
P\left(\sup_{\substack{\underline{s}\in[-S,S]^d\\ t=1,\dots,T}} w_{t,T}\, f\left(\underline{s}, \underline{X}^\star_{t,T}\right) > d_T^{\frac{2+\delta^2}{2(4+\delta)}}\right) \leq C\, d_T^{\frac{\delta-\delta^2}{2}} = o(1).
$$

This closes the proof. □

The next sequence of good sets we want to determine deals with the difference between bootstrap and real world expectation of the function g. The following lemma attends this matter.

Lemma A.5.
Under the validity of Assumptions 5,6 and 7, there exist subsets $(G_T)_{T\in\mathbb{N}}$ of Ω satisfying $P(G_T)\to 1$ as T tends to ∞ such that we have for $t=1,\dots,T$

$$
\sup_{1\leq t\leq T}\left|E^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) - Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right)\right| \leq d_T^{-\frac{\delta}{2(1+\delta)}}
$$

on G_T for the prevailing T.

Proof. For simplicity reasons, we will not meet the endpoint cases explicitly. Since the calculations do not differ in other terms that the choice of the index shift, consider the index as shifted, if necessary.

To begin with, it holds

$$\begin{aligned} EE^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) &= E\left(\frac{1}{2\,TD_T+1}\sum_{r=-TD_T}^{TD_T} g\left(\underline{X}_{t+r,T}, \underline{X}_{t+r,T}\right)\right) \\ &= \frac{1}{2\,TD_T+1}\sum_{r=-TD_T}^{TD_T} Eg\left(\underline{X}_{t+r,T}, \underline{X}_{t+r,T}\right) \\ &\leq C \end{aligned}$$

uniformly for all $t \in \{1, \ldots, T\}$. Thereby, we have

$$E\left|E^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) - Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right)\right| \leq EE^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) + Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right) \leq C$$

again uniformly over t. Next, we define

$$\begin{aligned} G_T &:= \left\{\omega \in \Omega \,\middle|\, \sup_{1\leq t\leq T} E\left|E^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) - Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right)\right| \leq d_T^{-\frac{\delta}{2(1+\delta)}}\right\} \\ &= \left\{\omega \in \Omega \,\middle|\, d_T^{\frac{\delta}{2(1+\delta)}} \sup_{1\leq t\leq T} E\left|E^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) - Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right)\right| \leq 1\right\}. \end{aligned}$$

This will be the sought-after good set. To verify that this is really the case, we consider the corresponding probability. As first step, we rewrite $P\left(\breve{A}_T\right)$ and obtain

$$\begin{aligned} P\left(G_T\right) &= 1 - P\left(\left(G_T\right)^C\right) \\ &= 1 - P\left(\left\{\omega \in \Omega \,\middle|\, d_T^{\frac{\delta}{2(1+\delta)}} \sup_{1\leq t\leq T} E\left|E^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) - Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right)\right| > 1\right\}\right) \\ &\geq 1 - E\left(d_T^{\frac{\delta}{2(1+\delta)}} \sup_{1\leq t\leq T} E\left|E^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) - Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right)\right|\right). \end{aligned} \tag{A.113}$$

To examine the expectation further, we make use of the calculations we did in the beginning on this proof. As a result, we have

$$E\left(d_T^{\frac{\delta}{2(1+\delta)}} \sup_{1\leq t\leq T} E\left|E^\star g\left(\underline{X}^\star_{t,T}, \underline{X}^\star_{t,T}\right) - Eg\left(\underline{X}_{t,T}, \underline{X}_{t,T}\right)\right|\right) = \mathcal{O}\left(d_T^{\frac{\delta}{2(1+\delta)}}\right).$$

Since $d_T^{\frac{\delta}{2(1+\delta)}}$ tends to 0 if T tends to ∞, it holds $P\left(G_T\right) \to 1$ as $T \to \infty$ according to equation (A.113). This finishes the proof. □

At this point, we turn our attention back to the findings of Section 3.4 and continue with the proof of Lemma 3.11.

Proof of Lemma 3.11. Since we want to establish an upper bound, we start by looking at the absolute value of the double sum, which, in turn, can be bounded by

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T} \\ \cdot \left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\right|. \tag{A.114}$$

Now we aim for relating the last covariance of (A.114) to the difference between $\underline{s}_1$ and $\underline{s}_2$ but without abandoning the capacity to be totaled. For that purpose, we mainly make use of the moment conditions given in Assumption 2 and Hoelder's inequality. While Assumption 5 would allow for the existence of higher moments, we keep the later arising exponents simpler by only relying on the lower-order moments. First, we split the covariance up inserting the truncated version of the companion process with truncation parameter $M := \lceil |h|/2 \rceil$ like in equation (2.13) as follows:

$$\begin{aligned}
&\left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&= \left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) \right.\right. \\
&\qquad\qquad - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right), \\
&\qquad\qquad\quad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\qquad + \left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), \right.\right. \\
&\qquad\qquad\quad f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) \\
&\qquad\qquad\qquad \left.\left. - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{A.115}$$

We examine only term I because term II can be treated analogously due to their similar building type. The first step is to rewrite the covariance as expectations. Then, it holds

$$
\begin{aligned}
\mathrm{I} \leq E \Bigg(\Bigg| & f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) \\
& - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg| \\
& \cdot \left|f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\Bigg) \\
& + E\Bigg|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) \\
& - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg| \\
& \cdot E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right| \\
=: {} & \mathrm{Ia} + \mathrm{Ib}.
\end{aligned} \tag{A.116}
$$

We move on with Ia and get with use of the Cauchy-Schwarz inequality

$$
\begin{aligned}
\mathrm{Ia} \leq \Bigg(E\Bigg| & f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) \\
& - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg|^2\Bigg)^{1/2} \\
& \cdot \left(E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|^2\right)^{1/2} \\
=: {} & \mathrm{Iaa} \cdot \mathrm{Iab}.
\end{aligned} \tag{A.117}
$$

Continuing again with the first factor, that is Iaa, we apply Hoelder's inequality to split it up as follows:

$$
\begin{aligned}
\mathrm{Iaa} \leq \Bigg(E\Bigg| & f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) \\
& - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg|^{\frac{2+\delta}{1+\delta}}\Bigg)^{\frac{1+\delta}{2+\delta}\cdot\frac{1}{2}} \\
& \cdot \Bigg(E\Bigg|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) \\
& - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\Bigg|^{2+\delta}\Bigg)^{\frac{1}{2+\delta}\cdot\frac{1}{2}} \\
=: {} & \mathrm{Iaaa} \cdot \mathrm{Iaab}.
\end{aligned} \tag{A.118}
$$

Playing on the Lipschitz condition 2.11 stated in Assumption 2, we can bound Iaaa via

$$\text{Iaaa} \leq C\left(E\left|\underline{\widetilde{X}}_0\left(\frac{t}{T}\right) - \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right|^{\frac{2+\delta}{1+\delta}}\right)^{\frac{1+\delta}{2(2+\delta)}} \leq C\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{1/2}.$$

On the other hand, Iaab originating in (A.118) can be bounded by a constant because only finite $(2+\delta)$-th absolute moments are required. Hence, combining those two upper bounds, we get

$$\text{Iaa} \leq C\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{1/2}.$$

For Iab in equation (A.117), we use Hoelder's inequality again and obtain

$$\begin{aligned}\text{Iab} \leq &\left(E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|^{\frac{2+\delta}{1+\delta}}\right)^{\frac{1+\delta}{2+\delta}\cdot\frac{1}{2}} \\ &\qquad\qquad \cdot\left(E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|^{2+\delta}\right)^{\frac{1}{2+\delta}\cdot\frac{1}{2}} \\ &=: \text{Iaba}\cdot\text{Iabb}.\end{aligned}$$

The first factor from above can be bounded using once again the fulfillment of the Lipschitz condition (2.11) due to Assumption 2:

$$\text{Iaba} \leq |\underline{s}_1 - \underline{s}_2|_1^{1/2}\left(E\left(g\left(\underline{\widetilde{X}}_h\left(\frac{t}{T}\right), \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)^{\frac{2+\delta}{1+\delta}}\right)^{\frac{1+\delta}{2(2+\delta)}} \leq C\,|\underline{s}_1 - \underline{s}_2|_1^{1/2},$$

whereas Iabb, like Iaab before, can be bounded by a constant $C < \infty$. This means we can bound Iab by $C\,|\underline{s}_1 - \underline{s}_2|_1^{1/2}$, and therefore we get

$$\text{Ia} \leq C\left(\sum_{|j|\geq M}\frac{B}{l(j)}\right)^{1/2}|\underline{s}_1 - \underline{s}_2|_1^{1/2}.$$

Now we have a look at the remaining subterm of equation (A.116), namely Ib. Since f fulfills the Lipschitz condition 2.11 stated in Assumption 2 and because of

$$\begin{aligned}&E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right| \\ &\leq E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right| \\ &\qquad\qquad + E\left|f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right| \\ &\leq C\sum_{|j|\geq M}\frac{B}{l(j)},\end{aligned}$$

we obtain

$$\mathrm{Ib} \leq C \sum_{|j| \geq M} \frac{B}{l(j)} \left| \underline{s}_1 - \underline{s}_2 \right|_1 .$$

Consequently, we have

$$\mathrm{I} \leq C \left(\left(\sum_{|j| \geq M} \frac{B}{l(j)} \right)^{1/2} \left| \underline{s}_1 - \underline{s}_2 \right|_1^{1/2} + \sum_{|j| \geq M} \frac{B}{l(j)} \left| \underline{s}_1 - \underline{s}_2 \right|_1 \right) .$$

We see that the bound consists of a sum of the same expression, where the first summand is taken to the power of 1/2. As is holds

$$\sum_{|j| \geq M} \frac{B}{l(j)} \leq \left(\sum_{|j| \geq M} \frac{B}{l(j)} \right)^{1/2} \cdot \left(\sum_{j \in \mathbb{Z}} \frac{B}{l(j)} \right)^{1/2} \leq C \left(\sum_{|j| \geq M} \frac{B}{l(j)} \right)^{1/2}$$

due to the capability of the sum to be totaled stated in Assumption 1 as well as

$$\left| \underline{s}_1 - \underline{s}_2 \right|_1 \leq \left| \underline{s}_1 - \underline{s}_2 \right|_1^{1/2} (2\, Sd)^{1/2} = C \left| \underline{s}_1 - \underline{s}_2 \right|_1^{1/2} ,$$

we can modify the bound to get

$$\mathrm{I} \leq C \left(\sum_{|j| \geq M} \frac{B}{l(j)} \right)^{1/2} \left| \underline{s}_1 - \underline{s}_2 \right|_1^{1/2} .$$

As explained earlier, we get the same upper bound for term II in (A.115). Therefore, we conclude

$$\begin{aligned} &\left| \mathrm{Cov} \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0 \left(\frac{t}{T} \right) \right) - f \left(\underline{s}_2, \underline{\widetilde{X}}_0 \left(\frac{t}{T} \right) \right), f \left(\underline{s}_1, \underline{\widetilde{X}}_h \left(\frac{t}{T} \right) \right) - f \left(\underline{s}_2, \underline{\widetilde{X}}_h \left(\frac{t}{T} \right) \right) \right) \right| \\ &\qquad \leq C \left(\sum_{|j| \geq M} \frac{B}{l(j)} \right)^{1/2} \left| \underline{s}_1 - \underline{s}_2 \right|_1^{1/2} . \end{aligned}$$

Hence, we can bound (A.114) by

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T \lceil (TD_T+1)/L_T \rceil + 1}^{L_T \lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \left(\sum_{|j| \geq M} \frac{B}{l(j)} \right)^{1/2} \left| \underline{s}_1 - \underline{s}_2 \right|_1^{1/2}$$

and obtain

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$\leq C_{DC}\, |\underline{s}_1 - \underline{s}_2|_1^{1/2}$$

due to Assumption 3, which terminates the proof. □

After this short side trip, we go back to the determination of the good sets. However, the next lemma is related to the one belonging to the previous proof as it turns the bootstrap variance into real world covariances of the exact form ready to apply Lemma 3.11 to.

Lemma A.6.
Assuming Assumptions 5 and 6 are fulfilled. Then, there exist subsets $(B_T)_{T\in\mathbb{N}}$ of Ω with $P(B_T) \to 1$ as $T \to \infty$ such that

$$\operatorname{Var}^\star\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}^\star_{t,T}\right) - f\left(\underline{s}_2, \underline{X}^\star_{t,T}\right)\right)\right)$$
$$= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$+ \mathcal{O}\left(\frac{L_T}{T^{1/3}}\right) + \mathcal{O}\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right) + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right)$$

holds on B_T for $\underline{s}_1, \underline{s}_2 \in [-S,S]^d$, where both $\underline{s}_1$ and $\underline{s}_2$ have no influence on the $\mathcal{O}$-terms.

Proof. The main aspect of the following proof will be the replacement of $\mathcal{O}_P$-terms by $\mathcal{O}$-terms with slightly modified rates. These exchanges will take place on certain subsets of Ω, whose intersection, namely B_T, satisfies the sought-after equality.

Similarly to the proof of Theorem 3.8, we start by dividing the sum in order to separate the endpoint indices the way only whole bootstrap blocks are kept in the middle sum. As in said proof, we assume T to be large enough so that $L_T < d_T^{-\frac{\delta}{2(1+\delta)}}$ holds. Thus, we obtain

$$\begin{aligned}
&\sum_{t=1}^{T} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right) \\
&= \sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right) \\
&\qquad + \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right) \\
&\qquad\qquad + \sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right).
\end{aligned}$$

This partition can, as before, be transferred to the bootstrap variance, since the blocks are independent. This leads to

$$\begin{aligned}
&\operatorname{Var}^\star\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right)\right) \\
&= \operatorname{Var}^\star\left(\sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right)\right) \\
&\qquad + \operatorname{Var}^\star\left(\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right)\right) \\
&\qquad\qquad + \operatorname{Var}^\star\left(\sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right)\right) \\
&=: R_T^I + V_T + R_T^{II}.
\end{aligned} \tag{A.119}$$

Before we turn our attention to the main sum V_T, we assure that the two rest terms R_T^I and R_T^I are negligible. In order to do so, we bound the former by

$$\begin{aligned}
&\sum_{t_1,t_2=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t_1,T}\, w_{t_2,T} \operatorname{Cov}^\star\left(f\left(\underline{s}_1, \underline{X}_{t_1,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t_1,T}^\star\right), f\left(\underline{s}_1, \underline{X}_{t_2,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t_2,T}^\star\right)\right) \\
&= \mathcal{O}_P\left(\left(TD_T + L_T\right)^2 d_T\right) \\
&= \mathcal{O}_P\left(d_T^{\frac{\delta}{2+\delta}}\right)
\end{aligned}$$

using the same calculations as in the aforementioned proof. Because of $L_T\lfloor T/L_T\rfloor \le T$, we obtain the same bound for R_T^{II}, that is $R_T^{II} = \mathcal{O}_P\left(d_T^{\frac{\delta}{2+\delta}}\right)$. These bounds form the first $\mathcal{O}_P$-terms we need to replace. Hence, we have to determine suitable subsets of Ω. Beginning with R_T^I, we define

$$B_T^I := \left\{ \omega \in \Omega \,\middle|\, R_T^I \leq d_T^{\frac{\delta}{2(1+\delta)}} \right\} = \left\{ \omega \in \Omega \,\middle|\, d_T^{-\frac{\delta}{2(1+\delta)}} R_T^I \leq 1 \right\}.$$

Now we verify that the probability of B_T^I tends to 1 when $T \to \infty$. Therefore, we establish a lower bound as follows:

$$\begin{aligned} P\left(B_T^I\right) &= 1 - P\left(\left(B_T^I\right)^C\right) \\ &= 1 - P\left(\left\{ \omega \in \Omega \,\middle|\, d_T^{-\frac{\delta}{2(1+\delta)}} R_T^I \leq 1 \right\}\right) \\ &\geq 1 - E\left(d_T^{-\frac{\delta}{2(1+\delta)}} R_T^I \right). \end{aligned} \tag{A.120}$$

The expectation in (A.120) can be bounded via

$$E\left(d_T^{\frac{\delta}{2(1+\delta)}} R_T^I \right) = \mathcal{O}\left(d_T^{-\frac{\delta}{2(1+\delta)} + \frac{\delta}{2+\delta}} \right) = \mathcal{O}\left(d_T^{\frac{\delta^2}{2(1+\delta)(2+\delta)}} \right).$$

Thus, inserting this rate into (A.120) we get $\lim_{T\to\infty} P\left(B_T^I\right) = 1$. Analogously, we define

$$B_T^{II} := \left\{ \omega \in \Omega \,\middle|\, R_T^{II} \leq d_T^{\frac{\delta}{2(1+\delta)}} \right\}$$

and obtain $\lim_{T\to\infty} P\left(B_T^{II}\right) = 1$ performing the same steps as above. At this point, we come back to the remaining sum of equation (A.119). Rewriting the variance as sums of covariances leads to

$$\begin{aligned} V_T = &\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \\ &\cdot \mathrm{Cov}^\star\left(f\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right) - f\left(\underline{s}_2, \underline{X}^\star_{tL_T+j,T}\right), f\left(\underline{s}_1, \underline{X}^\star_{tL_T+l,T}\right) - f\left(\underline{s}_2, \underline{X}^\star_{tL_T+l,T}\right)\right). \end{aligned}$$

Since the arguments of the covariances are differences, the expression from above can be split up further by

$$\begin{aligned} &\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}\; \mathrm{Cov}^\star\left(f\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right), f\left(\underline{s}_1, \underline{X}^\star_{tL_T+l,T}\right)\right) \\ &- \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}\; \mathrm{Cov}^\star\left(f\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right), f\left(\underline{s}_2, \underline{X}^\star_{tL_T+l,T}\right)\right) \\ &- \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}\; \mathrm{Cov}^\star\left(f\left(\underline{s}_2, \underline{X}^\star_{tL_T+j,T}\right), f\left(\underline{s}_1, \underline{X}^\star_{tL_T+l,T}\right)\right) \end{aligned}$$

$$+\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1}\sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}\ \operatorname{Cov}^\star\left(f\left(\underline{s}_2, \underline{X}^\star_{tL_T+j,T}\right), f\left(\underline{s}_2, \underline{X}^\star_{tL_T+l,T}\right)\right). \tag{A.121}$$

Now we remember the steps made in the proof of Theorem 3.8 to turn equation (A.121) into

$$\begin{aligned}
&\sum_{h=-(L_T-1)}^{L_T-1}\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\
&-\sum_{h=-(L_T-1)}^{L_T-1}\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\
&-\sum_{h=-(L_T-1)}^{L_T-1}\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \operatorname{Cov}\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\
&+\sum_{h=-(L_T-1)}^{L_T-1}\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \operatorname{Cov}\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\
&\qquad + R_T^{III} + R_T^{IV} + R_T^{V} + \mathcal{O}\left(L_T^{-1}\right) + \mathcal{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right)
\end{aligned} \tag{A.122}$$

with rest terms R_T^{III}, R_T^{IV} and R_T^{V}. Each of these rest terms are composed of four different rest terms belonging to the combinations of $\underline{s}_1$ and $\underline{s}_2$ used in the first arguments of the function f. In detail, we have $R_T^{III} := \sum_{i_1,i_2=1}^{2} R_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)$, $R_T^{IV} := \sum_{i_1,i_2=1}^{2} R_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)$ and $R_T^{V} := \sum_{i_1,i_2=1}^{2} R_T^{V}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)$. From the proof of Theorem 3.8, we know the related convergence rates of these twelve single rest terms. Since R_T^{III}, R_T^{IV} and R_T^{V} are defined by summation, we can carry over these rates and obtain $R_T^{III} = \mathcal{O}_P\left(\frac{L_T}{T^{\frac{3+\delta}{2(4+\delta)}}}\right)$, $R_T^{IV} = \mathcal{O}_P\left(L_T\, D_T^{\frac{3+\delta}{2(4+\delta)}}\right)$ and $R_T^{V} = \mathcal{O}_P\left(\frac{L_T}{TD_T}\right)$. Following the same argumentation, we perform the exchange of the terms for the dozen and transfer the results to the three main rest terms by intersecting the respective subsets of Ω. Before we start with the replacement of the $\mathcal{O}_P$-terms, we concentrate (A.122) back into

$$\begin{aligned}
&\sum_{h=-(L_T-1)}^{L_T-1}\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\\
&\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\
&\qquad + R_T^{III} + R_T^{IV} + R_T^{V} + \mathcal{O}\left(L_T^{-1}\right) + \mathcal{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right).
\end{aligned} \tag{A.123}$$

Now we look at the rest terms. For all $i_1, i_2 \in \{1,2\}$, we have

$$
\begin{aligned}
R_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = \Bigg| & \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \\
& \cdot \Bigg(\Bigg(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{X}_{tL_T+j+r,T}\right) f\left(\underline{s}_{i_2}, \underline{X}_{tL_T+l+r,T}\right) \\
& - \frac{1}{\left(2\,TD_T+1\right)^2} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{X}_{tL_T+j+r,T}\right) \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{X}_{tL_T+l+k,T}\right) \Bigg) \\
& - \Bigg(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \\
& \cdot f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\right) \\
& - \frac{1}{\left(2\,TD_T+1\right)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \right. \\
& \left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right)\right)\Bigg)\Bigg) \Bigg| .
\end{aligned}
$$

Then we define

$$
B_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) := \left\{ \omega \in \Omega \,\middle|\, R_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq \frac{L_T}{T^{1/3}} \right\} = \left\{ \omega \in \Omega \,\middle|\, \frac{T^{1/3}}{L_T} R_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq 1 \right\}.
$$

As before, the next step is to check for convergence of the probability of the above-defined B_T^{III}. It holds

$$
P\left(B_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) \geq 1 - E\left(\frac{T^{1/3}}{L_T} R_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right).
$$

Focusing on the expectation, we obtain

$$
E\left(\frac{T^{1/3}}{L_T} R_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) = \mathcal{O}\left(\frac{T^{1/3}}{L_T} \cdot \frac{L_T}{T^{\frac{3+\delta}{2(4+\delta)}}}\right) = \mathcal{O}\left(T^{-\frac{1+\delta}{6(4+\delta)}}\right)
$$

and conclude $\lim_{T\to\infty} P\left(B_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) = 1$ for all $i_1, i_2 \in \{1,2\}$. Accordingly, setting

$$
B_T^{III} := \bigcap_{i_1, i_2 \in \{1,2\}} B_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)
$$

we get $\lim_{T\to\infty} P\left(B_T^{III}\right) = 1$. Moving on to R_T^{IV}, it holds for each combination of i and j belonging to $\{1,2\}$

$$
\begin{aligned}
R_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = \Bigg| & \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{l=\max\{1,1-h\}}^{\min\{L_T, L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T} \\
& \cdot \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right)\right. \\
& \qquad \cdot f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \\
& - \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right)\right. \\
& \qquad \left.\left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j+k}{T}\right)\right)\right)\right) \\
& - \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right)\right. \\
& \qquad \cdot f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\right) \\
& - \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right)\right. \\
& \qquad \left.\left.\left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\right)\right)\right)\right) \Bigg| .
\end{aligned}
$$

Thus, we define

$$
\begin{aligned}
B_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) &:= \left\{ \omega \in \Omega \,\middle|\, R_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}} \right\} \\
&= \left\{ \omega \in \Omega \,\middle|\, \left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right)^{-1} R_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq 1 \right\}
\end{aligned}
$$

and obtain for the belonging probability

$$
P\left(B_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) \geq 1 - E\left(\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right)^{-1} R_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right).
$$

The expectation fulfills

$$
E\left(\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right)^{-1} R_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) = \mathcal{O}\left(L_T^{-1} D_T^{-\frac{3+\delta^2}{2(4+\delta)}}\, L_T\, D_T^{\frac{3+\delta}{2(4+\delta)}}\right) = \mathcal{O}\left(D_T^{\frac{\delta-\delta^2}{2(4+\delta)}}\right)
$$

leading to $\lim_{T\to\infty} P\left(B_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) = 1$ for all $i_1, i_2 \in \{1, 2\}$. Hence, for

$$B_T^{IV} := \bigcap_{i_1,i_2\in\{1,2\}} B_T^{IV}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right),$$

we can conclude $\lim_{T\to\infty} P\left(B_T^{IV}\right)=1$. Lastly, we have

$$\begin{aligned}
R_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right) = \Bigg| \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{l=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} & w_{tL_T+j+h,T}\, w_{tL_T+j,T} \\
\cdot \Bigg(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right) & \\
\cdot f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\right) & \\
- \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right)\right. & \\
\left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\right)\right) & \\
- \operatorname{Cov}\left(f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h}\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j}\left(\frac{tL_T+j}{T}\right)\right)\right)\Bigg)\Bigg|
\end{aligned}$$

for every pair (i,j) with $i_1,i_2\in\{1,2\}$. Following once again the same pattern as before we define

$$\begin{aligned}
B_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right) &:= \left\{\omega\in\Omega \,\middle|\, R_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right) \le \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right\} \\
&= \left\{\omega\in\Omega \,\middle|\, \frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\, R_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right) \le 1\right\}
\end{aligned}$$

and have a look at the probability

$$P\left(B_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right)\right) \ge 1 - E\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\, R_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right)\right).$$

The expectation satisfies

$$E\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\, R_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right)\right) = \mathcal{O}\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\cdot\frac{L_T}{TD_T}\right) = \mathcal{O}\left((TD_T)^{-\delta/4}\right)$$

in this case and, consequently, it holds $\lim_{T\to\infty} P\left(B_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right)\right)=1$. Defining

$$B_T^{V} := \bigcap_{i_1,i_2\in\{1,2\}} B_T^{V}\left(\underline{s}_{i_1},\underline{s}_{i_2}\right)$$

leads to the desired convergence $\lim_{T\to\infty} P\left(B_T^V\right) = 1$.

The next step will be to combine the main subsets we defined before in a suitable way. To this end, we define the intersection of all of them by B_T, that is

$$B_T := B_T^I \cap B_T^{II} \cap B_T^{III} \cap B_T^{IV} \cap B_T^V. \tag{A.124}$$

This newly-defined subset of Ω is the set we are now working on since it holds

$$P\left(B_T\right) \longrightarrow 1$$

as $T \to \infty$ due to the construction of B_T. As the remaining $\mathcal{O}$-terms of (A.123) are not restricted on a certain subset of Ω, they stay valid on B_T. So, we have for every $\omega \in B_T$

$$\begin{aligned}
&\operatorname{Var}^\star\left(\sum_{t=1}^{\lfloor T/L_T\rfloor-1} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right)\right)\right)\\
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\\
&\quad\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\
&\qquad + \mathcal{O}\left(d_T^{\frac{\delta}{2(1+\delta)}}\right) + \mathcal{O}\left(\frac{L_T}{T^{1/3}}\right) + \mathcal{O}\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right) + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right)\\
&\qquad + \mathcal{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right).
\end{aligned}$$

Now we want to investigate our terms further to see if we can incorporate certain terms into others to minimize the number of different $\mathcal{O}$-terms in the end. First, we compare the rates originated in the determination of B_T^I and B_T^{II} with the next-to-last $\mathcal{O}$-term, that is $\mathcal{O}\left(L_T^{-1}\right)$. Since the rates belonging to B_T^I and B_T^{II} are $\mathcal{O}\left(d_T^{\frac{\delta}{2(1+\delta)}}\right)$ in both cases, we only need to do the comparison once. Following Assumption 6, for sufficiently large T it holds $L_T^{-1} > d_T^{\frac{\delta}{2(1+\delta)}}$. Consequently, we have

$$\mathcal{O}\left(d_T^{\frac{\delta}{2(1+\delta)}}\right) \subseteq \mathcal{O}\left(L_T^{-1}\right).$$

Next, we incorporate $\mathcal{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right)$ into $\mathcal{O}\left(L_T^{-1}\right)$. To justify this, we verify

$$L_T^{-1} \geq \left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}$$

for T large enough. This is equivalent to

$$T^{\frac{\delta}{1+\delta}} \geq L_T^{\frac{1+2\delta}{1+\delta}}$$

for large T. The left-hand side can be bounded from beneath by $d_T^{-\frac{\delta}{1+\delta}}$ and the right-hand side from above by $d_T^{-\frac{\delta(1+2\delta)}{2(1+\delta)^2}}$ for T above a certain threshold. Hence, we have to check for

$$\frac{\delta}{1+\delta} \geq \frac{\delta(1+2\delta)}{2(1+\delta)^2}$$

since d_T is smaller than 1 for all T. We can simplify the equation from above and get

$$1 \geq \frac{1+2\delta}{2+2\delta}$$

as a new constraint, which is fulfilled for every $\delta \in (0,1)$. Thus, we have

$$\mathcal{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right) \subseteq \mathcal{O}\left(L_T^{-1}\right).$$

In conclusion, we have for every $\omega \in B_T$

$$\begin{aligned}
&\operatorname{Var}^\star\left(\sum_{t=1}^{\lfloor T/L_T\rfloor -1} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}^\star_{t,T}\right) - f\left(\underline{s}_2, \underline{X}^\star_{t,T}\right)\right)\right)\\
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\\
&\quad \cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\
&\qquad + \mathcal{O}\left(\frac{L_T}{T^{1/3}}\right) + \mathcal{O}\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right) + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right),
\end{aligned}$$

which ends the proof. □

The last sequence of good sets we are going to determine in this subsection will find its use in the last part of the tightness proof.

Lemma A.7.
Provided the validity of Assumptions 6 and 7, there exist subsets $(K_T)_{T\in\mathbb{N}}$ of Ω, on which it holds

$$\sum_{t=0}^{\lfloor T/L_T\rfloor -1} E^\star\left(\left|L^\star_{t,T}\right|_\infty^{1-\delta}\left|L^\star_{t,T}\right|_{\mathrm{Lip}}^{1+\delta}\right) \leq L_T^{1+\delta}$$

for the respective T with

$$L^\star_{t,T}\left(\underline{s}\right) := \sum_{j=1}^{L_T} w_{tL_T+j,T}\, f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T}\right)$$

for $\underline{s} \in [-S,S]^d$. Furthermore, it holds $P\left(K_T\right) \to 1$ as $T \to \infty$.

Proof. First of all, it holds

$$\sum_{t=0}^{\lfloor T/L_T \rfloor -1} E^\star \left(\left| L^\star_{t,T} \right|_\infty^{1-\delta} \left| L^\star_{t,T} \right|_{\text{Lip}}^{1+\delta} \right) \leq \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(E^\star \left| L^\star_{t,T} \right|_\infty^{2(1-\delta)} \right)^{1/2} \left(E^\star \left| L^\star_{t,T} \right|_{\text{Lip}}^{2(1+\delta)} \right)^{1/2} \tag{A.125}$$

by the application of Hoelder's inequality.

As in the proof of Lemma A.6, we want to determine the sought-after subset K_T by modifying a $\mathcal{O}_P$-term, which bounds the right-hand side of (A.125). To this end, we need to establish said term first. Therefore, consider the expectation of the expression in question, that is

$$E \left(\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(E^\star \left| L^\star_{t,T} \right|_\infty^{2(1-\delta)} \right)^{1/2} \left(E^\star \left| L^\star_{t,T} \right|_{\text{Lip}}^{2(1+\delta)} \right)^{1/2} \right). \tag{A.126}$$

With Jensen's inequlity plus a de novo use of Hoelder's inequlity, we can bound equation (A.126) by

$$\begin{aligned}
&\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(E \left(E^\star \left| L^\star_{t,T} \right|_\infty^{2(1-\delta)} E^\star \left| L^\star_{t,T} \right|_{\text{Lip}}^{2(1+\delta)} \right) \right)^{1/2} \\
\leq &\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(\left(E E^\star \left| L^\star_{t,T} \right|_\infty^{4+\delta} \right)^{\frac{2(1-\delta)}{4+\delta}} \left(E E^\star \left| L^\star_{t,T} \right|_{\text{Lip}}^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \right)^{\frac{2+3\delta}{4+\delta}} \right)^{1/2} \\
= &\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(E E^\star \left| L^\star_{t,T} \right|_\infty^{4+\delta} \right)^{\frac{1-\delta}{4+\delta}} \left(E E^\star \left| L^\star_{t,T} \right|_{\text{Lip}}^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \right)^{\frac{2+3\delta}{2(4+\delta)}}.
\end{aligned} \tag{A.127}$$

Now we examine both the infinity and the Lipschitz seminorm combined with the bootstrap expectation as a preliminary point, before we investigate the expression as a whole. Starting with the infinity norm, it holds

$$\begin{aligned}
&E^\star \left| L^\star_{t,T} \right|_\infty^{4+\delta} \\
&= E^\star \left(\sup_{\underline{s} \in [-S,S]^d} \left| \sum_{j=1}^{L_T} w_{tL_T+j,T} \, f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T} \right) \right| \right)^{4+\delta} \\
&= E^\star \left(\sup_{\underline{s} \in [-S,S]^d} \left| \sum_{j=1}^{L_T} w_{tL_T+j,T} \left(f\left(\underline{s}, \underline{X}^\star_{tL_T+j,T} \right) - f\left(\underline{0}, \underline{X}^\star_{tL_T+j,T} \right) + f\left(\underline{0}, \underline{X}^\star_{tL_T+j,T} \right) \right) \right| \right)^{4+\delta} \\
&\leq E^\star \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(S d g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T} \right) + \left| f\left(\underline{0}, \underline{X}^\star_{tL_T+j,T} \right) \right| \right) \right)^{4+\delta}
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{j=1}^{L_T} w_{tL_T+j,T} \left(Sd\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right) + \left| f\left(\underline{0}, \underline{X}^\star_{tL_T+j,T}\right)\right| \right) \right\|_{4+\delta,\star}^{4+\delta} \\
&\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(Sd \left\| g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{4+\delta,\star} + \left\| f\left(\underline{0}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{4+\delta,\star} \right) \right)^{4+\delta} \\
&\leq C \left(\left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{4+\delta,\star} \right)^{4+\delta} \right. \\
&\qquad\qquad \left. + \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| f\left(\underline{0}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{4+\delta,\star} \right)^{4+\delta} \right) \qquad \text{(A.128)}
\end{aligned}
$$

due to the Lipschitz condition stated in Assumption 2. Subsequently, we turn our attention to the Lipschitz seminorm in (A.127) and obtain

$$
\begin{aligned}
&E^\star \left| L^\star_{t,T} \right|_{\mathrm{Lip}}^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \\
&= E^\star \left(\sup_{\substack{\underline{s}_1 \underline{s}_2 \in [-S,S]^d \\ \underline{s}_1 \neq \underline{s}_2}} \frac{\left| \sum_{j=1}^{L_T} w_{tL_T+j,T} \left(f\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right) - f\left(\underline{s}_2, \underline{X}^\star_{tL_T+j,T}\right)\right)\right|}{\left|\underline{s}_1 - \underline{s}_2\right|_1} \right)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \\
&\leq E^\star \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right) \right)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \\
&= \left\| \sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right) \right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star}^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \\
&\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star} \right)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \qquad \text{(A.129)}
\end{aligned}
$$

using the Lipschitz condition (2.11) once more. At this point, we return to equation (A.127) and insert the upper bounds derived in (A.128) and (A.129), which leads to

$$
\begin{aligned}
C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} &\left(E \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{4+\delta,\star} \right)^{4+\delta} \right. \\
&\qquad \left. + E \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| f\left(\underline{0}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{4+\delta,\star} \right)^{4+\delta} \right)^{\frac{1-\delta}{4+\delta}} \\
&\qquad \cdot \left(E \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star} \right)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \right)^{\frac{2+3\delta}{2(4+\delta)}}
\end{aligned}
$$

$$\leq C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \Bigg(\Bigg(\Bigg(E \Bigg(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \Bigg)^{4+\delta} \Bigg)^{\frac{1-\delta}{4+\delta}}$$

$$+ \Bigg(E \Bigg(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| f\left(\underline{0}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \Bigg)^{4+\delta} \Bigg)^{\frac{1-\delta}{4+\delta}} \Bigg)$$

$$\cdot \Bigg(E \Bigg(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star} \Bigg)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \Bigg)^{\frac{2+3\delta}{2(4+\delta)}}$$

$$=: C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} (\mathrm{I} + \mathrm{II}) \cdot \mathrm{III} \tag{A.130}$$

instead of (A.127) to go on with. Now we investigate the three newly defined terms one by one beginning with the second. We have

$$\begin{aligned} \mathrm{II} &= \left\| \sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| f\left(\underline{0}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right\|_{4+\delta}^{1-\delta} \\ &\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| \left\| f\left(\underline{0}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right\|_{4+\delta} \right)^{1-\delta}. \end{aligned} \tag{A.131}$$

Focusing on the nested norms, we assume w.l.o.g. $tL_T + j \notin EP$ and get

$$\begin{aligned} \left\| \left\| f\left(\underline{0}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right\|_{4+\delta} &= \left(E \left\| f\left(\underline{0}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star}^{4+\delta} \right)^{\frac{1}{4-\delta}} \\ &= \left(E \left(\frac{1}{2\,TD_T + 1} \sum_{r=-TD_T}^{TD_T} \left| f\left(\underline{0}, \underline{X}_{tL_T+j,T} \right) \right|^{4+\delta} \right) \right)^{\frac{1}{4-\delta}} \\ &\leq C_1 \end{aligned} \tag{A.132}$$

since f possesses finite absolute moments of order $4+\delta$. However, for $tL_T + j \in EP$ we would have obtained the same result because the boundedness is not affected by the summation. Due to the same reason, we restrict ourselves to the sheer consideration of non-endpoint cases for the remaining part of the proof. Inserted in (A.131), the recently established bound from above simplifies the upper bound for term II, to wit

$$\mathrm{II} \leq C_1 \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{1-\delta}. \tag{A.133}$$

Proceeding with the first subterm of (A.132), it holds

$$\mathrm{I} \leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right\|_{4+\delta} \right)^{1-\delta} .$$

In contrast to our proceedings above, we postpone the investigation of the nested norms. At first, we consider only the inner part, namely the bootstrap $\mathcal{L}^{4+\delta}$-norm of g. In order to play on part (ii) of Assumption 7, we need to transform the arguments of g. Therefore, we get by adding a suitable self-canceling difference

$$\begin{aligned}
&\left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \\
&= \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \\
&\qquad - \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \\
&\qquad + \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} .
\end{aligned} \tag{A.134}$$

Now we return to the nested norms and replace the inner norm with equation (A.134). In doing so, we obtain

$$\begin{aligned}
&\left\| \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right\|_{4+\delta} \\
&\leq \left\| \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right. \\
&\qquad - \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left. \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \right\|_{4+\delta} \\
&\qquad + \left\| \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \right.\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left. \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \right\|_{4+\delta} \\
&=: \mathrm{Ia} + \mathrm{Ib}.
\end{aligned} \tag{A.135}$$

Again, we continue step-by-step and investigate subterm Ia in the first place. It holds

$$
\begin{aligned}
\mathrm{Ia} = &\left(E \left| \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{X}_{tL_T+j,T}, \underline{X}_{tL_T+j,T} \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \right.\right. \\
&\qquad - \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left.\left. \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \\
\leq &\left(E \left| \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{X}_{tL_T+j,T}, \underline{X}_{tL_T+j,T} \right) \right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. -g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \\
= &\left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} E \left| g\left(\underline{X}_{tL_T+j+r,T}, \underline{X}_{tL_T+j+r,T} \right) \right.\right. \\
&\qquad \left.\left. -g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \\
= &\left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left\| g\left(\underline{X}_{tL_T+j+r,T}, \underline{X}_{tL_T+j+r,T} \right) \right.\right. \\
&\qquad \left.\left. -g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right), \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right\|_{4+\delta}^{4+\delta} \right)^{\frac{1}{4+\delta}} \\
\leq &\left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left(\frac{C_g}{T} \right)^{4+\delta} \right)^{\frac{1}{4+\delta}} \\
\leq &\frac{C_g}{T}
\end{aligned} \tag{A.136}
$$

with the use of part (iii) of Assumption 7. The remaining summand of equation(A.135), Ib, can be bounded by a constant C_2 analogously to (A.132) because of the finite $(4+\delta)$-th absolute moments of g. Together with equation (A.136), this leads to

$$
\mathrm{I} \leq C \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1-\delta} . \tag{A.137}
$$

At this point, only term III of equation (A.130) is left to be examined. Because said term has the same building type as term I, we can repeat the belonging procedure to obtain

$$\left(E \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T} \right) \right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star} \right)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \right)^{\frac{2+3\delta}{2(4+\delta)}}$$
$$= \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| \left\| g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T} \right) \right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star} \right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \right)^{1+\delta}$$
$$\leq C \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1+\delta}. \tag{A.138}$$

Joining equations (A.133), (A.137) and (A.138), we obtain

$$C \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(\left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1-\delta} + \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{1-\delta} \right)$$
$$\cdot \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1+\delta} \tag{A.139}$$

as an upper bound for equation (A.130). In addition to that, we concentrate on the product and get via expanding

$$\left(\left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1-\delta} + \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{1-\delta} \right) \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1+\delta}$$
$$\leq C \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1-\delta} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1} + 1 \right) \right)^{1+\delta}$$
$$\leq C \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{2}. \tag{A.140}$$

Inserting (A.140) back into (A.139) yields

$$C \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^2 \leq C \, \frac{d_T^{-1}}{L_T} L_T^2 \, d_T = C L_T$$

as new bound from above for equation (A.130). Thus, we have shown

$$\sum_{t=0}^{\lfloor T/L_T \rfloor - 1} E^\star \left(\left| L^\star_{t,T} \right|_\infty^{1-\delta} \left| L^\star_{t,T} \right|_{\mathrm{Lip}}^{1+\delta} \right) = \mathcal{O}_P \left(L_T \right).$$

Based on this, we consider

$$P\left(\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^\star\left(\left|L_{t,T}^\star\right|_\infty^{1-\delta}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{1+\delta}\right) > L_T\right).$$

Comparably to the proof of Lemma A.6, we modify the lower bound in the equation above in order to classify said probability as $o(1)$. in preference to $\mathcal{O}(1)$. Therefore, consider the following set

$$K_T := \left\{\omega\in\Omega \,\middle|\, \sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^\star\left(\left|L_{t,T}^\star\right|_\infty^{1-\delta}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{1+\delta}\right) \le L_T^{1+\delta}\right\}.$$

depending on T. Next, we show that $P(K_T)$ tends to 1 while $T\to\infty$. We have

$$\begin{aligned}
P(K_T) &= 1 - P\left(K_T^C\right)\\
&= 1 - P\left(\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^\star\left(\left|L_{t,T}^\star\right|_\infty^{1-\delta}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{1+\delta}\right) > L_T^{1+\delta}\right)\\
&= 1 - P\left(L_T^{-(1+\delta)}\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^\star\left(\left|L_{t,T}^\star\right|_\infty^{1-\delta}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{1+\delta}\right) > 1\right)\\
&\ge 1 - E\left(L_T^{-(1+\delta)}\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^\star\left(\left|L_{t,T}^\star\right|_\infty^{1-\delta}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{1+\delta}\right)\right).
\end{aligned}$$

Now we focus on the expectation and obtain using the results from the beginning of this proof

$$E\left(L_T^{-(1+\delta)}\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^\star\left(\left|L_{t,T}^\star\right|_\infty^{1-\delta}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{1+\delta}\right)\right) = \mathcal{O}\left(L_T^{-(1+\delta)}L_T\right) = \mathcal{O}\left(L_T^{-\delta}\right),$$

which tends to 0 for $T\to\infty$. Consequently, it holds $\lim_{T\to\infty} P(K_T) = 1$. This terminates the proof. □

With the lemma above, all needed good sets are established, and we can directly move on to the demonstration of the tightness result in Lemma 3.12.

Proof of Lemma 3.12. The first step will be to define the subsets $(\Omega_T)_{T\in\mathbb{N}}$ of Ω. Remember the subsets A_T, G_T, B_T and K_T established for each T in Lemmata A.4, A.5, A.6 and A.7, respectively. These are the subsets we need to conduct the proof. Therefore, we set

$$\Omega_T := A_T\cap G_T\cap B_T\cap K_T,$$

and thus it holds $\lim_{T\to\infty} P(\Omega_T) = 1$. The following steps of the proof will take place on this specific subset depending on T.

Now we move on using again the proof performed in Arcones and Yu (1994) as a guideline. We start by splitting the left-hand side of (3.5) up the way that we get one sum containing the indices of whole independent blocks and a second one containing the remaining indices. Thereby, we ignore the limits. This leads to

$$\begin{aligned}
&P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\lambda\right)\\
&\leq P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\frac{\lambda}{2}\right)\\
&\quad+P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\frac{\lambda}{2}\right). \qquad (A.141)
\end{aligned}$$

Using Markov's inequality and Lemma A.5, we can bound the second sum of (A.141) as follows:

$$\begin{aligned}
&P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\frac{\lambda}{2}\right)\\
&\leq\frac{2}{\lambda}\,E^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|\right)\\
&\leq\frac{2}{\lambda}\,E^\star\left(\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right|\right.\right.\\
&\qquad\qquad\left.\left.+\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}E^\star\left|f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)\right|\right)\right)\\
&\leq\frac{2}{\lambda}\,E^\star\left(\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(r\,g\left(\underline{X}^\star_{t,T},\underline{X}^\star_{t,T}\right)+r\,E^\star g\left(\underline{X}^\star_{t,T},\underline{X}^\star_{t,T}\right)\right)\right)\\
&=\frac{2\,r}{\lambda}\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left\{E^\star g\left(\underline{X}^\star_{t,T},\underline{X}^\star_{t,T}\right)+E^\star g\left(\underline{X}^\star_{t,T},\underline{X}^\star_{t,T}\right)\right)\\
&=\frac{4\,r}{\lambda}\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(Eg\left(\underline{X}_{t,T},\underline{X}_{t,T}\right)+\mathcal{O}\left(d_T^{-\frac{\delta}{2(1+\delta)}}\right)\right)\\
&\leq L_T\,d_T^{1/2}\,\mathcal{O}\left(d_T^{-\frac{\delta}{2(1+\delta)}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}\left(d_T^{-\frac{2\delta}{2(1+\delta)}+\frac{1}{2}}\right) \\
&= \mathcal{O}\left(d_T^{\frac{1-\delta}{2(1+\delta)}}\right),
\end{aligned}
$$

which tends to 0 as $T \to \infty$ leaving us with the first sum of (A.141) to deal with. We start by introducing some further notation comparable to the one introduced in the proof of Lemma 2.18. Considering again $\underline{s}, \underline{s}_1, \underline{s}_2 \in [-S, S]^d$, we define $\nu_T^\star(\underline{s})$ as well as $\nu_T^\star(\underline{s}_1, \underline{s}_2)$ by

$$
\nu_T^\star(\underline{s}) := \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \sum_{j=1}^{L_T} w_{tL_T+j,T}\, \bar{f}^\star\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right) \quad \text{and} \quad \nu_T^\star\left(\underline{s}_1, \underline{s}_2\right) := \nu_T^\star\left(\underline{s}_1\right) - \nu_T^\star\left(\underline{s}_2\right),
$$

respectively. We make use of the same chaining argument as in the proof of Lemma 2.18 containing the decreasing sequence

$$
r_k := r\, 2^{-k}
$$

for $k = 0, \ldots, k_T$. Again, existence and order of r_{k_T} will be specified later on. In the following, we will use the same notation regarding the index sets $(\mathcal{F}_k)_{k=0}^{k_T}$, the maps $(\pi_k)_{k=0}^{k_T}$ and the packing number $D\left(u, [-S, S]^d, \rho\right)$. Because of this, we do not repeat all of the definitions and inequalities in this place. Having this setup in mind, we can turn the left over sum of (A.141) into

$$
\begin{aligned}
&P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1, \underline{X}^\star_{t,T}\right) - \bar{f}^\star\left(\underline{s}_2, \underline{X}^\star_{t,T}\right)\right)\right| > \frac{\lambda}{2}\right) \\
&= P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\sum_{j=1}^{L_T} w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right) - \bar{f}^\star\left(\underline{s}_2, \underline{X}^\star_{tL_T+j,T}\right)\right)\right| > \frac{\lambda}{2}\right) \\
&= P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\nu_T^\star\left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{2}\right) \\
&\leq P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\nu_T^\star\left(\underline{s}_1\right) - \nu_T^\star\left(\underline{s}_2\right) - \nu_T^\star\left(\pi_{k_T}\,\underline{s}_1\right) + \nu_T^\star\left(\pi_{k_T}\,\underline{s}_2\right)\right| > \frac{\lambda}{6}\right) \\
&\qquad\qquad + P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\nu_T^\star\left(\pi_0\,\underline{s}_1\right) - \nu_T^\star\left(\pi_0\,\underline{s}_2\right)\right| > \frac{\lambda}{6}\right) \\
&\quad + P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}} \left|\sum_{k=1}^{k_T} \nu_T^\star\left(\pi_k\,\underline{s}_1\right) - \nu_T^\star\left(\pi_{k-1}\,\underline{s}_1\right) - \nu_T^\star\left(\pi_k\,\underline{s}_2\right) + \nu_T^\star\left(\pi_{k-1}\,\underline{s}_2\right)\right| > \frac{\lambda}{6}\right)
\end{aligned}
$$

$$\leq P^\star\left(2\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{6}\right)+P^\star\left(2\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}}\left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{6}\right)$$

$$+P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}}\left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{6}\right)$$

$$=: \mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{A.142}$$

Similarly to the proof of Lemma 2.18, we use Bernstein's inequality for the discussion of terms II and III starting with II. In order to do so, we are in need to establish an upper bound for the inner sum of

$$\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)=\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\sum_{j=1}^{L_T}w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right)\right).$$

Looking at the absolute value, we get

$$\begin{aligned}\left|\sum_{j=1}^{L_T}w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right)\right)\right| &\leq 4\,L_T\,d_T^{\frac{2+\delta^2}{2(4+\delta)}}\\ &< C\,d_T^{-\frac{\delta}{2(1+\delta)}+\frac{2+\delta^2}{2(4+\delta)}}\\ &= C\,d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\end{aligned} \tag{A.143}$$

for T larger than a suitable $\bar{T}$ due to Assumption 6. For Bernstein's inequality, the second moment condition to deal with concerns the bootstrap variance of $\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)$. With help of Lemmata 3.11 and A.6, we obtain

$$\begin{aligned}&\mathrm{Var}^\star\left(\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right)\\ &=\mathrm{Var}^\star\left(\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\sum_{j=1}^{L_T}w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right)\right)\right)\\ &=\mathrm{Var}^\star\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor}w_{t,T}\left(f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right)\\ &=\sum_{h=-(L_T-1)}^{L_T-1}\sum_{t=L_T\lceil(TD_T+1)/L_T\rceil+1}^{L_T\lfloor(T-TD_T)/L_T\rfloor}w_{t+h,T}\,w_{t,T}\\ &\quad\cdot\mathrm{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right),f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\\ &\qquad+\mathcal{O}\left(\frac{L_T}{T^{1/3}}\right)+\mathcal{O}\left(L_T\,D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right)+\mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right)+\mathcal{O}\left(L_T^{-1}\right)\end{aligned}$$

$$\leq C_{DC} \left| \underline{s}_1 - \underline{s}_2 \right|_1^{1/2} + \mathcal{O}\left(\frac{L_T}{T^{1/3}}\right) + \mathcal{O}\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right) + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right). \tag{A.144}$$

We aim for transforming these upper bounds in expressions using the above-defined $(r_k)_{k=0}^{k_T}$ with k given by the outer sum in term II. To this end, we start by establishing a relation between the ratios of the upper bounds in (A.143) and (A.144) and r_{k_T} as r_{k_T} can be easily replaced by r_k for every desired k later. Hence, we work with

$$\max\left\{\frac{L_T}{T^{1/3}}, L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}, \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}, L_T^{-1}, d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\right\}. \tag{A.145}$$

The first step is to compare the ratios originating in the bootstrap variance with the upper bound of the inner sum in order to minimize the number of arguments in the maximum above. Using Assumption 6, we get for the first ratio of (A.144)

$$\frac{L_T}{T^{1/3}} < d_T^{-\frac{\delta}{2(1+\delta)}+\frac{1}{3}} = d_T^{\frac{2-\delta}{6(1+\delta)}}$$

for $T > \bar{T}$. To compare this result with the last argument of (A.145), we can go on with the comparison of the exponents. We are willing to show

$$d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}} > d_T^{\frac{2-\delta}{6(1+\delta)}}. \tag{A.146}$$

Therefore, we verify

$$\frac{2-\delta}{6(1+\delta)} > \frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}, \tag{A.147}$$

which can be simplified notably by canceling. Subsequent regrouping leads finally to $2 + 4\delta - \delta^2 - 3\delta^3 > 0$ instead of (A.147), which is satisfied by any $\delta \in (0, 1)$. Thus, equation (A.146) holds true for $T > \bar{T}$, and we can turn (A.145) into

$$\max\left\{L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}, \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}, L_T^{-1}, d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\right\} \tag{A.148}$$

for T larger than $\bar{T}$. We propose now both a lower and an upper bound for r_{k_T}, and, afterwards, we will show the reasons why our choice is compatible with the conditions we have and hence the existence of a suitable r_{k_T} whose index is integer-valued. Thus, for $T > \bar{T}$ define

$$\left(\max\left\{L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}, \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}, L_T^{-1}, d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\right\}\right)^{\frac{(1+\delta)(2+\delta)(4+\delta)}{\delta^2(\delta-\delta^2)}} \leq r_{k_T} \leq L_T^{-\frac{1+\delta}{\delta^2}}. \tag{A.149}$$

What has to be verified is, as explained in the proof of Lemma 2.18 yet, the fact that the lower bound in (A.149) is not higher than the upper bound for our parameter choices.

For this purpose, we examine all four arguments of the maximum separately starting with the third. We aim to show

$$L_T^{-\frac{(1+\delta)(2+\delta)(4+\delta)}{\delta^2(\delta-\delta^2)}} < L_T^{-\frac{1+\delta}{\delta^2}}. \tag{A.150}$$

This inequality can be simplified by canceling, and we consider

$$L_T^{-\frac{(2+\delta)(4+\delta)}{(\delta-\delta^2)}} < L_T^{-1} \tag{A.151}$$

instead. Due to the same base, equation (A.151) is equivalent to

$$1 < \frac{(2+\delta)(4+\delta)}{(\delta-\delta^2)}. \tag{A.152}$$

Because the denominator on the right-hand side of (A.152) is smaller than 1 for $\delta \in (0,1)$, whereas the numerator is larger than 1 for the same choice of δ, equation (A.152) holds true for every allowed δ. In consequence, the same applies to (A.150). Now we move on to the first argument of the maximum in (A.149). In place of

$$\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right)^{\frac{(1+\delta)(2+\delta)(4+\delta)}{\delta^2(\delta-\delta^2)}} < L_T^{-\frac{1+\delta}{\delta^2}}, \tag{A.153}$$

it is enough to have a look at

$$\left(L_T\, D_T^{\frac{3+\delta^2}{2(4+\delta)}}\right)^{\frac{(2+\delta)(4+\delta)}{\delta-\delta^2}} < L_T^{-1}. \tag{A.154}$$

Since the left- and right-hand side do not have the same base, a direct comparison of the exponents, as before, is not possible. Therefore, we start by regrouping equation (A.154), which yields

$$D_T^{\frac{(2+\delta)\left(3+\delta^2\right)}{2(\delta-\delta^2)}} < L_T^{-\frac{8+7\delta}{\delta-\delta^2}}.$$

Moreover, this new inequality can be reduced to

$$D_T^{\frac{(2+\delta)\left(3+\delta^2\right)}{2}} < L_T^{-(8+7\delta)}. \tag{A.155}$$

In order to verify equation (A.155), we establish an upper bound for the left-hand side and a bound from beneath for the right-hand side of (A.155). Firstly, it holds

$$D_T^{\frac{(2+\delta)\left(3+\delta^2\right)}{2}} \le \left(d_T^{1-\frac{1}{2+\delta}}\right)^{\frac{(2+\delta)\left(3+\delta^2\right)}{2}} = d_T^{\frac{(1+\delta)\left(3+\delta^2\right)}{2}} \tag{A.156}$$

because of Assumption 6. On the other hand, the same assumption guarantees for

$$L_T^{-(8+7\delta)} > d_T^{\frac{\delta(8+7\delta)}{2(1+\delta)}} \tag{A.157}$$

for $T > \bar{T}$. Hence, we combine the bounds derived in (A.156) and (A.157) and verify

$$\frac{\delta(8+7\delta)}{1+\delta} \leq (1+\delta) \cdot \left(3+\delta^2\right) \tag{A.158}$$

in lieu of equation (A.155). This is possible due to the same base of the bounds. Equation (A.158) is equivalent to $3 - 2\delta - 3\delta^2 + 2\delta^3 + \delta^4 \geq 0$, which is fulfilled for every $\delta \in (0,1)$. Hence, equation (A.153) holds true for $T > \bar{T}$. Therefore, we can proceed with the maximum's second argument in (A.149) and have a look at the following inequality

$$\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right)^{\frac{(1+\delta)(2+\delta)(4+\delta)}{\delta^2(\delta-\delta^2)}} < L_T^{-\frac{1+\delta}{\delta^2}}, \tag{A.159}$$

which we aim to show for at least $T > \bar{T}$. At first, we cancel the exponents again to get

$$\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right)^{\frac{(2+\delta)(4+\delta)}{\delta-\delta^2}} < L_T^{-1}. \tag{A.160}$$

Rewriting (A.160) leads to

$$L_T^{\frac{8+7\delta}{\delta-\delta^2}} < (TD_T)^{\frac{4-\delta}{4} \cdot \frac{(2+\delta)(4+\delta)}{\delta-\delta^2}}.$$

Because of multiple similarities on both sides regarding the exponent, we can simplify the above-stated inequality by canceling suitable terms and obtain

$$L_T^{8+7\delta} < (TD_T)^{\frac{(2+\delta)\left(16-\delta^2\right)}{4}} \tag{A.161}$$

to move on with. Next, we bound the two sides of equation (A.161). First, we can bound the left-hand side via Assumption 6 for $T > \bar{T}$ as follows:

$$L_T^{8+7\delta} < d_T^{-\frac{\delta(8+7\delta)}{2(1+\delta)}}. \tag{A.162}$$

In addition to that, we use the same assumption to bound the right-hand side of (A.161) from beneath, to wit

$$(TD_T)^{\frac{(2+\delta)\left(16-\delta^2\right)}{4}} \geq d_T^{-\frac{\delta}{2+\delta} \cdot \frac{(2+\delta)\left(16-\delta^2\right)}{4}} = d_T^{-\frac{\delta\left(16-\delta^2\right)}{4}}. \tag{A.163}$$

A closer look at (A.162) and (A.163) reveals that a comparison of the exponents is enough to verify equation (A.161). For that purpose, we consider

$$\frac{\delta\,(8+7\delta)}{2(1+\delta)} \leq \frac{\delta\,(16-\delta^2)}{4},$$

which can be reduced to $\delta + \delta^2 \leq 2$. Since the last inequality is fulfilled for every $\delta \in (0,1)$, equation (A.161), and thus equation (A.159) as well, holds true for $T > \bar{T}$. Moving on to the remaining term in (A.149), that is $d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}$, note that it holds $d_T < L_T^{-\frac{2(1+\delta)}{\delta}}$ for $T > \bar{T}$. Thus, if we can guarantee for

$$\left(L_T^{-\frac{2(1+\delta)}{\delta}\cdot\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\right)^{\frac{(1+\delta)(2+\delta)(4+\delta)}{\delta^2(\delta-\delta^2)}} \leq L_T^{-\frac{1+\delta}{\delta^2}}, \tag{A.164}$$

we can conclude

$$\left(d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\right)^{\frac{(1+\delta)(2+\delta)(4+\delta)}{\delta^2(\delta-\delta^2)}} < L_T^{-\frac{1+\delta}{\delta^2}}. \tag{A.165}$$

Returning to equation (A.164), we simplify the inequality in question and get

$$\left(L_T^{-\frac{2-2\delta+\delta^3}{\delta(4+\delta)}}\right)^{\frac{(2+\delta)(4+\delta)}{\delta-\delta^2}} \leq L_T^{-1}.$$

Moving on, we have to compare only the exponents once again. To be precise, we need to check

$$\frac{2-2\delta+\delta}{\delta(4+\delta)}\cdot\frac{(2+\delta)(4+\delta)}{\delta-\delta^2} \geq 1.$$

The inequality above is equivalent to $2-2\delta+\delta^3 \geq \delta^2-\delta^3$, which is satisfied by every $\delta \in (0,1)$. Hence, this holds for all equivalent inequalities as well and we can eventually deduce the sought-after relation in (A.165) for $T > \bar{T}$.

Consequently, (A.149) presents a way to bound r_{k_T} in order to ensure its existence while following our special conditions imposed on the parameters. The specific choice of the upper bound in (A.149) will become relevant during the examination of term I of equation (A.142).

Returning to (A.144), in our calculations so far we neglected the difference between $\underline{s}_1$ and $\underline{s}_2$. As we wish to bound the bootstrap variance of $\nu_T^\star(\underline{s}_1,\underline{s}_2)$, we need to find a suitable bound for the said difference likewise. Since we are in the case where $\rho(\underline{s}_1,\underline{s}_2) \leq 3\, r_k$, again with k inherent to the outer sum in II, holds, we have a convenient bound by construction. Summing up, we get

$$\operatorname{Var}^\star\left(\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right) \leq C\left(r_k^{1/2}+r_{k_T}^{\frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)}}\right) \leq C\left(r_k^{1/2}+r_k^{\frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)}}\right) \tag{A.166}$$

for $T > \bar{T}$ as it holds $r_{k_T} < r_k$ for all $k = 0,1,\ldots,k_T$. As in the previous calculations, we want to condensate the last sum in (A.166) by identifying the smaller exponent because for r small enough, $r_k < 1$ holds for all $k = 0,1,\ldots,k_T$. This puts

$$\frac{1}{2} - \frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)} > 0$$

into consideration. The above-stated inequality can be rewritten as

$$(1+\delta)(2+\delta)(4+\delta) > 2\delta^2\left(\delta-\delta^2\right), \tag{A.167}$$

which is clearly satisfied by $\delta \in (0,1)$, because for this choice, the left-hand side is larger than 1 and the other side, as opposed to this, smaller than 1. Thus, instead of (A.166), we work with

$$\mathrm{Var}^\star\left(\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right) \le C\, r_k^{\frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)}} \tag{A.168}$$

for $T > \bar{T}$ and r small enough. After having determined the bounds to be used in Bernstein's inequality, we turn our attention to λ and, as in the proof of Lemma 2.18, define a sequence $(\lambda_k)_{k\in\mathbb{N}}$ satisfying

$$2\sum_{k\in\mathbb{N}}\lambda_k \le \frac{\lambda}{6}$$

for r sufficiently small. In addition to that and in order to be more precisely, we suppose

$$\lambda_k := r_k^{\frac{\delta^2\left(\delta-\delta^2\right)}{4(1+\delta)(2+\delta)(4+\delta)}} \vee \left(\frac{4}{\bar{C}}\log\left(D(k)\right) r_k^{\frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)}}\right)^{1/2}$$

for $k=1,\dots,k_T$ and a finite constant $\bar{C}>0$ which will be specified further during the upcoming calculations. Again, $D(k)=\mathcal{O}\left(r_k^{-d}\right)$ ensures the summability of $(\lambda_k)_{k=1}^{k_T}$ with T tending to ∞. Now we return to term II of equation (A.142) and use both the definition of $(\lambda_k)_{k=1}^{k_T}$ and Bernstein's inequality with the previously established upper bounds in (A.143) and (A.168) to get

$$\begin{aligned}
&\limsup_{T\to\infty} P^\star\left(2\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}}\left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{6}\right)\\
&\le \limsup_{T\to\infty} P^\star\left(2\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}}\left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right| > 2\sum_{k=1}^{k_T}\lambda_k\right)\\
&\le \limsup_{T\to\infty}\sum_{k=1}^{k_T} P^\star\left(\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}}\left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right| > \lambda_k\right)\\
&\le \limsup_{T\to\infty} 2\sum_{k=1}^{k_T} D(k)D(k-1)\exp\left(-\frac{1}{2}\cdot\frac{\lambda_k^2}{C_1\, r_k^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}} + C_2\, d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\lambda_k/3}\right)\\
&\le \limsup_{T\to\infty} 2\sum_{k=1}^{k_T}\exp\left(2\log(D(k)) - \frac{1}{2}\cdot\frac{\lambda_k^2}{C_1\, r_k^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}} + C_2\, r_{k_T}^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}}\lambda}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} \exp\left(2\log(D(k)) - \bar{C} \frac{\lambda_k^2}{r_k^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}}} \right) \\
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2} \cdot \frac{\lambda_k^2}{r_k^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}}} \right) \\
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2}\, r_k^{\frac{\delta^2\left(\delta-\delta^2\right)}{2(1+\delta)(2+\delta)(4+\delta)} - \frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)}} \right) \\
&\leq 2 \limsup_{T\to\infty} \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2}\, r_k^{-\frac{\delta^2\left(\delta-\delta^2\right)}{2(1+\delta)(2+\delta)(4+\delta)}} \right) \\
&\leq 2 \sum_{k\in\mathbb{N}} \exp\left(-C\, r^{-\frac{\delta^2\left(\delta-\delta^2\right)}{2(1+\delta)(2+\delta)(4+\delta)}} \right) \\
&\xrightarrow[r\to 0]{} 0.
\end{aligned}
$$

As term III of (A.142) is of the same building type as term II, we can proceed analogously. The bound established in (A.143) stays true, and because of $r = r_0$, the bound for the variance, (A.166), is valid with $k = 0$. This leads to

$$
\begin{aligned}
&\limsup_{T\to\infty} P^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0 \\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r}} \left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{6} \right) \\
&\leq \limsup_{T\to\infty} 2\, D^2(0) \exp\left(-\frac{1}{2} \cdot \frac{(\lambda/6)^2}{C_1\, r^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}} + C_2\, d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\lambda/18} \right) \\
&\leq 2\left(D\left(r, [-S,S]^d, \rho\right)\right)^2 \exp\left(-\frac{1}{2} \cdot \frac{1}{C_1\, r^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}} + C_2\, r^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}}} \right) \\
&\leq 2\left(\frac{2\,Sd}{r}+1\right)^{2d} \exp\left(-\frac{1}{2} \cdot \frac{1}{C\, r^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}}} \right) \\
&\leq \mathcal{O}\left(r^{-2d}\right) \exp\left(-C\, r^{-\frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)}} \right) \\
&\xrightarrow[r\to 0]{} 0.
\end{aligned}
$$

Now term I of (A.142) is left, and, as in the proof of Lemma 2.18, we aim for making use of a symmetrization lemma. Therefore, the following steps will be close to those performed in the aforesaid proof. First, we rewrite term I by dividing both sides by 2, so it holds

$$\mathrm{I} = P^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \nu_T^\star \left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{12} \right).$$

The next step will be to introduce some further notation, which is based on the counterpart in the proof of Lemma 2.18, that is

$$L_{t,T}^\star \left(\underline{s}\right) := \sum_{j=1}^{L_T} w_{tL_T+j,T}\, f\left(\underline{s}, \underline{X}_{tL_T+j,T}^\star\right)$$

and

$$L_{t,T}^{\star,0}\left(\underline{s}\right) := \zeta_t\, L_{t,T}^\star\left(\underline{s}\right),$$

where $(\zeta_t)_{t=0}^{\lfloor T/L_T \rfloor -1}$ are again i.i.d. Rademacher variables, but this time, they are independent of $(k_t)_{t=0}^{\lfloor T/L_T \rfloor -1}$ used in Algorithm 3.1. Due to

$$P^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \nu_T^\star \left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{12} \right) \leq \frac{12\, E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \nu_T^\star \left(\underline{s}_1, \underline{s}_2\right)\right| \right)}{\lambda},$$

it suffices, comparable to the proof of Lemma 2.18, to look at

$$E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \nu_T^\star \left(\underline{s}_1, \underline{s}_2\right)\right| \right),$$

for which we obtain

$$\begin{aligned}
&E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \nu_T^\star \left(\underline{s}_1, \underline{s}_2\right)\right| \right) \\
&= E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^\star\left(\underline{s}_1\right) - E^\star L_{t,T}^\star\left(\underline{s}_1\right) - L_{t,T}^\star\left(\underline{s}_2\right) + E^\star L_{t,T}^\star\left(\underline{s}_2\right)\right)\right| \right) \\
&\leq 2\, E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^{\star,0}\left(\underline{s}_1\right) - L_{t,T}^{\star,0}\left(\underline{s}_2\right)\right)\right| \right) \qquad \text{(A.169)}
\end{aligned}$$

using the previously introduced notation and the same symmetrization lemma as in the proof of Lemma 2.18. Defining

$$\hat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) := \left(\sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(L^{\star}_{t,T}\left(\underline{s}_1\right) - L^{\star}_{t,T}\left(\underline{s}_2\right)\right)^2 \right)^{1/2} \tag{A.170}$$

for $\underline{s}_1, \underline{s}_2 \in [-S, S]^d$, Hoeffding's inequality gives

$$\begin{aligned} &P^{\star}\left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} L^{\star,0}_{t,T}\left(\underline{s}_1\right) - L^{\star,0}_{t,T}\left(\underline{s}_2\right) \right| > \hat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) \eta \;\middle|\; L^{\star}_{1,T}, \ldots, L^{\star}_{L_T \lfloor T/L_T \rfloor, T} \right) \\ &\leq 2 \exp\left(- \frac{\hat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right)^2 \eta^2}{2 \sum_{t=1}^{L_T \lfloor T/L_T \rfloor} \left(L^{\star}_{t,T}\left(\underline{s}_1\right) - L^{\star}_{t,T}\left(\underline{s}_2\right)\right)^2} \right) \\ &= 2 \exp\left(- \frac{\eta^2}{2} \right) \end{aligned}$$

for some $\eta > 0$. Thus, $\sum_{t=0}^{\lfloor T/L_T \rfloor - 1} L^{\star,0}_{t,T}$ possesses sub-Gaussian increments conditionally on $L^{\star}_{0,T}, \ldots, L^{\star}_{\lfloor T/L_T \rfloor - 1, T}$. We continue by establishing an upper bound for the difference in (A.170) and get

$$\begin{aligned} \left(L^{\star}_{t,T}\left(\underline{s}_1\right) - L^{\star}_{t,T}\left(\underline{s}_2\right)\right)^2 &\leq \left(\left| L^{\star}_{t,T}\left(\underline{s}_1\right) \right| + \left| L^{\star}_{t,T}\left(\underline{s}_2\right) \right| \right)^{1-\delta} \left| L^{\star}_{t,T}\left(\underline{s}_1\right) - L^{\star}_{t,T}\left(\underline{s}_2\right) \right|^{1+\delta} \\ &\leq 2^{1-\delta} \; \left| L^{\star}_{t,T} \right|_{\infty}^{1-\delta} \left| L^{\star}_{t,T} \right|_{\mathrm{Lip}}^{1+\delta} \rho\left(\underline{s}_1, \underline{s}_2\right)^{1+\delta} \end{aligned} \tag{A.171}$$

for $\underline{s}_1, \underline{s}_2 \in [-S, S]^d$. In order to create a more suitable semimetric during the further course of this proof, we define

$$Q_T := 2^{\frac{1-\delta}{2}} \left(\sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left| L^{\star}_{t,T} \right|_{\infty}^{1-\delta} \left| L^{\star}_{t,T} \right|_{\mathrm{Lip}}^{1+\delta} \right)^{1/2} .$$

Owing to the use of parts of the upper bound, we already see a connection between Q_T and $\hat{\rho}_{T,2}$. Furthermore, this connection helps us to construct the aforementioned semimetric:

$$\hat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) \leq Q_T \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{1+\delta}{2}} =: \breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right).$$

The newly defined $\breve{\rho}_T$ is in fact a semimetric as it holds

$$\breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) \leq Q_T \left(\rho\left(\underline{s}_1, \underline{s}_3\right) + \rho\left(\underline{s}_3, \underline{s}_2\right)\right)^{\frac{1+\delta}{2}} \leq \breve{\rho}_T\left(\underline{s}_1, \underline{s}_3\right) + \breve{\rho}_T\left(\underline{s}_3, \underline{s}_2\right).$$

Now we come back to (A.169). Considering the conditional expectation

$$E^{\star}\left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(L^{\star,0}_{t,T}\left(\underline{s}_1\right) - L^{\star,0}_{t,T}\left(\underline{s}_2\right)\right) \right| \;\middle|\; L^{\star}_{1,T}, \ldots, L^{\star}_{L_T \lfloor T/L_T \rfloor, T} \right),$$

we get using Corollary 2.2.8 of van der Vaart and Wellner (2000), a maximal inequality for sub-Gaussian processes,

$$
\begin{aligned}
&E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \le r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \right) \right| \Bigg| L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star \right) \\
&= E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \breve{\rho}(\underline{s}_1,\underline{s}_2) \le Q_T r_{k_T}^{\frac{1+\delta}{2}}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \right) \right| \Bigg| L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star \right) \\
&\le C \left(\int_0^{Q_T r_{k_T}^{\frac{1+\delta}{2}}} \left(\log D\left(u, [-S,S]^d, \breve{\rho}_T\right) \right)^{1/2} du \right).
\end{aligned}
$$

Next, we want to replace the packing number by its upper bound in order to have an explicit function to integrate. Using the relation between $\breve{\rho}_T$ and ρ_T we have

$$
D\left(u, [-S,S]^d, \breve{\rho}_T\right) = D\left(\left(\frac{u}{Q_T} \right)^{\frac{2}{1+\delta}}, [-S,S]^d, \rho \right) \le \left(\frac{2Sd}{\left(\frac{u}{Q_T} \right)^{\frac{2}{1+\delta}}} + 1 \right)^d,
$$

which leads to

$$
\begin{aligned}
&E^\star \left(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2) \le r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \right) \right| \right) \\
&\le C E^\star \left(\int_0^{Q_T r_{k_T}^{\frac{1+\delta}{2}}} \left(\log \left(\frac{2Sd}{\left(\frac{u}{Q_T} \right)^{\frac{2}{1+\delta}}} + 1 \right)^d \right)^{1/2} du \right) \\
&= C E^\star Q_T \int_0^{r_{k_T}^{\frac{1+\delta}{2}}} \left(\log \left(\frac{2Sd}{u^{\frac{2}{1+\delta}}} + 1 \right)^d \right)^{1/2} du. \qquad \text{(A.172)}
\end{aligned}
$$

At this point, we focus on the bootstrap expectation in (A.172). It holds

$$
E^\star Q_T \le C_1 \left(\sum_{t=0}^{\lfloor T/L_T \rfloor -1} E^\star \left(\left| L_{t,T}^\star \right|_\infty^{1-\delta} \left| L_{t,T}^\star \right|_{\mathrm{Lip}}^{1+\delta} \right) \right)^{1/2} \qquad \text{(A.173)}
$$

using Jensen's inequality. By Lemma A.7, we can bound the right-hand side of equation (A.173) by $C_1 L_T^{\frac{1+\delta}{2}}$. Returning to (A.172), we obtain by taking said upper bound into account

$$\limsup_{T\to\infty} E^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(L_{t,T}^{\star,0}(\underline{s}_1)-L_{t,T}^{\star,0}(\underline{s}_2)\right)\right|\right)$$
$$\le \limsup_{T\to\infty} C L_T^{\frac{1+\delta}{2}}\int_0^{r_{k_T}^{\frac{1+\delta}{2}}}\left(\log\left(\frac{2Sd}{u^{\frac{2}{1+\delta}}}+1\right)^d\right)^{1/2} du. \quad \text{(A.174)}$$

For the integral in (A.174), we get remembering the upper bound of r_{k_T} in (A.149)

$$\begin{aligned}\int_0^{r_{k_T}^{\frac{1+\delta}{2}}}\left(\log\left(\frac{2Sd}{u^{\frac{2}{1+\delta}}}+1\right)^d\right)^{1/2} du &\le \int_0^{r_{k_T}^{\frac{1+\delta}{2}}} u^{-\frac{1}{1+\delta}}\, du\\ &\le C\left[u^{\frac{\delta}{1+\delta}}\right]_0^{r_{k_T}^{\frac{1+\delta}{2}}}\\ &= C\, r_{k_T}^{\delta/2}\\ &\le C L_T^{-\frac{1+\delta}{\delta^2}\cdot\frac{\delta}{2}}\\ &= C L_T^{-\frac{1+\delta}{2\delta}} \end{aligned} \quad \text{(A.175)}$$

for $T>\bar{T}$ as $\log(x+1)\le x$ for $x>0$. Combining (A.174) and (A.175), it holds

$$\begin{aligned}&\limsup_{T\to\infty} E^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(L_{t,T}^{\star,0}(\underline{s}_1)-L_{t,T}^{\star,0}(\underline{s}_2)\right)\right|\right)\\ &\le \limsup_{T\to\infty} C L_T^{\frac{1+\delta}{2}} L_T^{-\frac{1+\delta}{2\delta}}\\ &= \limsup_{T\to\infty} C L_T^{\frac{1+\delta}{2}\left(1-\frac{1}{\delta}\right)}\\ &= 0\end{aligned}$$

since δ^{-1} is larger than 1 for $\delta\in(0,1)$. This terminates the proof. □

At this point, the only result left to show is the the main one. Thus, we proceed with the proof appurtenant to the bootstrap FCLT.

Proof of Theorem 3.13. As explained in the proof of Theorem 2.19, according to Theorem 1.5.4 of van der Vaart and Wellner (2000), convergence of the fidis and asymptotic tightness have to be shown. Besides, Theorem 1.5.7 of van der Vaart and Wellner (2000) is the reason why it suffices to show tightness in the sense of Lemma 3.12. Theorem 3.9 provides the convergence of the fidis and joint with Lemma 3.12, we obtain process convergence. Since both mentioned results are true in P-probability, the process convergence holds in P-probability as well. Again, Addendum 1.5.8 of van der Vaart and Wellner (2000) allows us to deduce the continuity of the sample path of the limiting process. □

The last proof closes the first half of this section addressing the unbounded case regarding the function f. The second is the counterpart for bounded functions f. Consequently, the structure remains the same as before. However, some results become obsolete.

A.2.3. Proofs of Section 3.5

As its bounded counterpart, this subsection contains the results corresponding to the real world processes, expectations and covariances, which are needed to proof the bootstrap results belonging to the CLT. As before, we start with the proof addressing the covariance of products of the function $\bar{f}$.

Proof of Lemma 3.14. This proof will be the analogon to the proof of Lemma 3.4 for a bounded function f.

(i) We start again with the determination of the truncation parameter, which turns out to be the same as in the previously mentioned proof due to the same constraints concerning the indices. Hence, we have $M := \left\lceil \frac{t_1-t_2}{2} \right\rceil$. Instead of following the scheme of the proof of version (a) of Lemma 2.11, we now take the one belonging to version (b) as a guideline. As in the first part proof of Lemma 3.4, we get

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\leq \Bigg|\operatorname{Cov}\Bigg(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \\
&\qquad\qquad \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\Bigg)\Bigg| \\
&\qquad + \Bigg|\operatorname{Cov}\Bigg(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \\
&\qquad\qquad\qquad\qquad \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\Bigg)\Bigg| \\
&=: \mathrm{I} + \mathrm{II}
\end{aligned} \tag{A.176}$$

by inserting the truncated version of the companion process with $\bar{f}_M$ as defined in (3.3). Therefore, the examination of term I is again sufficient, which can be bounded by

$$\begin{aligned}
&\Bigg|\operatorname{Cov}\Bigg(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \\
&\qquad\qquad \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) \left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\Bigg)\Bigg| \\
&\qquad + \Bigg|\operatorname{Cov}\Bigg(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), \\
&\qquad\qquad \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right) \left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right) - \bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right)\Bigg)\Bigg| \\
&=: \mathrm{Ia} + \mathrm{Ib}.
\end{aligned} \tag{A.177}$$

Because of the similar structure, we focus on Ia and obtain

$$\begin{aligned}
&\left|E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right)\right| \\
&\qquad\qquad\qquad\qquad + E\left|\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right)\right| \\
&\qquad\qquad\qquad\qquad\qquad \cdot E\left|\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\le 2\,C_f\,E\left|\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right| \\
&\le C\sum_{|j|\ge M}\frac{B}{l(j)}
\end{aligned}$$

as an upper bound. This result holds for the second subterm of equation (A.177), Ib, as well, which leads to

$$\mathrm{I}\le C\sum_{|j|\ge M}\frac{B}{l(j)}.$$

The very same bound is also valid for term II of (A.176). Consequently, we get

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|\le C\sum_{|j|\ge M}\frac{B}{l(j)}.$$

As last step, we use equation (2.14) to obtain

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\le C\sum_{|j|\ge M}\frac{|j|^{1+\tilde{\delta}}B}{|j|^{1+\tilde{\delta}}l(j)} \\
&\le \frac{C}{(t_1-t_2)^{1+\tilde{\delta}}}\sum_{|j|\ge M}\frac{|j|^{1+\tilde{\delta}}B}{l(j)} \\
&\le \frac{C_{Cov,2i,b}}{(t_1-t_2)^{1+\tilde{\delta}}}
\end{aligned}$$

for some $C_{Cov,2i,b}<\infty$.

(ii) Since we have $t_1,t_2\in\mathbb{N}$ with $t_1<t_2$, it holds

$$t_1+t_2>t_2>t_1>0.$$

Again, we examine two different cases, to wit

$$(1)\ t_1 > \frac{t_2}{2} \quad \text{and} \quad (2)\ t_1 \leq \frac{t_2}{2}.$$

Afterwards, as seen in the proof of Lemma 3.4, we join the arisen constants.

a) In the same manner as in part (i) of this proof, we benefit from the insertion of the truncated versions of the companion process. Similar to the real world counterpart, we work with $M := \left\lceil \frac{t_2 - t_1}{2} \right\rceil$ as truncation parameter in this case. Following the same argumentation as in the proof belonging to Lemma 3.4, we skip the next lines and proceed immediately with

$$C \sum_{|j| \geq M} \frac{B}{l(j)}$$

as bound from above to the expression in question. Rewriting said bound leads to

$$C \sum_{|j| \geq M} \frac{|j|^{1+\tilde{\delta}} B}{|j|^{1+\tilde{\delta}} l(j)}, \tag{A.178}$$

and with the use of equation (2.14), we bound (A.178) by

$$\frac{C}{(t_2 - t_1)^{1+\tilde{\delta}}} \sum_{|j| \geq M} \frac{|j|^{1+\tilde{\delta}} B}{l(j)} \leq \frac{C_{Cov,2ii1,b}}{(t_2 - t_1)^{1+\tilde{\delta}}}$$

for some finite constant $C_{Cov,2ii1,b} > 0$.

b) Once again, we benefit from the truncated versions of the companion processes, but this time with $M := \lceil t_1/2 \rceil$. Comparable to above, we consider directly the upper bound for equation (A.178). Here, we obtain

$$\frac{C}{t_1^{1+\tilde{\delta}}} \sum_{|j| \geq M} \frac{|j|^{1+\tilde{\delta}} B}{l(j)} \leq \frac{C_{Cov,2ii2,b}}{t_1^{1+\tilde{\delta}}}$$

for some positive constant $C_{Cov,2ii2,b} < \infty$.

To join the two new constants $C_{Cov,2ii1}$ and $C_{Cov,2ii2}$ from above, we set

$$C_{Cov,2ii,b} := \max \left\{ C_{Cov,2ii1,b}, C_{Cov,2ii2,b} \right\}.$$

At this point, both this part and the whole proof are brought to an end.

□

Now we move on to the weighted sums of expectations, which portrays the first auxiliary result of this subsection. As the function f is bounded, the proof can be eased a lot in comparison to Lemma A.1.

Lemma A.8.
Suppose Assumptions 4 for $k = 1$, 3 and 6 hold true. Then, we have for all $\underline{s} \in [-S,S]^d$, $t_1, t_2 \in \mathbb{Z}$ and $u \in [0,1]$

(i)

$$\frac{1}{(2TD_T+1)^2} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right) \leq \frac{C_{sum,1,b}}{TD_T},$$

(ii)

$$\frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right) \leq \frac{C_{sum,2,b}}{TD_T}$$

and

(iii)

$$\frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \sum_{m=-TD_T}^{TD_T} {}''E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+m}(u)\right)\right) \leq \frac{C_{sum,3,b}}{TD_T}.$$

Here, $\bar{f}$ is defined as in (2.15) and denotes the centered version of f, whereas $C_{sum,1,b}$ as well as $C_{sum,2,b}$ and $C_{sum,3,b}$ are each finite positive constants not dependent either on t_1, t_2 or on $\underline{s}$ and u.

Proof. To prove the first part, we will follow the lines of the proof of Lemma A.1, the real world counterpart to this lemma, but when it comes to the second and third part, we can abridge the proof by explaining these cases by part (i).

(i) W.l.o.g. let $t_1 \geq t_2$ again. Due to stationarity, it holds

$$\begin{aligned}
&\sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \left|E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right)\right| \\
&= \sum_{\substack{l=-2TD_T \\ l \neq t_2-t_1}}^{2TD_T} (2TD_T+1-|l|) \left|\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1-t_2+l}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right)\right| \\
&\qquad + (2TD_T+1-|t_2-t_1|) \left|\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0(u)\right)\right)\right|
\end{aligned}$$

$$=: \mathrm{I} + \mathrm{II}$$

as seen in the proof of Lemma A.1. By using the very same argumentation as in said proof, we obtain $\mathrm{II} \leq C_{sum,1,ba}\, TD_T$ for some positive constant $C_{sum,1,ba} < \infty$. Hence, we continue with term I and the use of version (b) of Lemma 2.11. Similarly to the proof of the counterpart with unbounded f, we get

$$\mathrm{I} \leq \sum_{\substack{l=-2TD_T \\ l \neq t_2 - t_1}}^{2TD_T} (2TD_T + 1 - |l|) \frac{C_{Cov}}{|t_1 - t_2 + l|^{1+\tilde{\delta}}} \leq C_{sum,1,bb}\, TD_T$$

with finite constant $C_{sum,1,bb} > 0$. Hence, it holds

$$\frac{1}{(2TD_T+1)^2} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \left| E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right)\right| = \frac{C_{sum,1,b}}{TD_T}.$$

(ii) Again, w.l.o.g. assume $t_1 \geq t_2$. Moreover, we set $v := t_1 - t_2$. Similar to the proof appertaining to part (ii) of Lemma A.1, stationarity of the companion process leads to

$$\begin{aligned}
&\left| \frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \right. \\
&\qquad \left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right) \right| \\
&\leq \left| \frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l=-TD_T \\ l \neq r}}^{TD_T} \sum_{\substack{k=-TD_T \\ k \neq r}}^{TD_T} \right. \\
&\qquad \left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right| \\
&\quad + \left| \frac{1}{(2TD_T+1)^3} \sum_{\substack{l,r=-TD_T \\ l=r}}^{TD_T} \sum_{\substack{k=-TD_T \\ k \neq r}}^{TD_T} E\left(\bar{f}^2\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right| \\
&\quad + \left| \frac{1}{(2TD_T+1)^3} \sum_{\substack{r,k=-TD_T \\ r=k}}^{TD_T} \sum_{l=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}^2\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right)\right) \right| \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Since the function f is bounded, both terms II and III can be bounded by $\frac{C_{sum,2,ba}}{TD_T}$ with some constant $0 < C_{sum,2,ba} < \infty$ leaving us with the examination of term I as in the proof of part (ii) of Lemma A.1. Because in said proof, the following steps do not rely directly on the finiteness of certain moments of f but only while using Lemma 3.4, we can relegate to those combined with Lemma 3.14 here. This finishes the proof of part (ii).

(iii) As seen in the previous part, we can make use of the steps performed in the proof of the related Lemma A.1. This extends to the take-over of all steps excluding those dealing immediately with the boundedness of the $(4+\delta)$-th absolute moments instead of f itself. At these points, we either use the boundedness of the function directly or follow the argumentation of part (ii) of this proof. This closes part (iii) and hence the proof as a whole.

□

Now, as in Subsection A.2.1, we turn our attention to the transition from the locally stationary process to the companion one when products are involved:

Lemma A.9.
Under Assumptions 4 for $k = 1$ and 3, we have for all $t_1, t_2 \in \{1, \ldots, T\}$ and $s \in [-S, S]^d$

$$f\left(\underline{s}, \underline{X}_{t_1,T}\right) f\left(\underline{s}, \underline{X}_{t_2,T}\right) = f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right) + \mathcal{O}_P\left(T^{-1}\right).$$

Here, the $\mathcal{O}_P$-term depends neither on the choice of t_1 and t_2 nor on $\underline{s}$.

Proof. As in the proof appertaining to Lemma A.2, we use the closeness between the process $\left(\underline{X}_{t,T}\right)$ and its companion process $\left(\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)$ and get with the help of Lemma 2.6

$$\begin{aligned}
&E\left|f\left(\underline{s}, \underline{X}_{t_1,T}\right) f\left(\underline{s}, \underline{X}_{t_2,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)\right| \\
&\leq E\left|\left(f\left(\underline{s}, \underline{X}_{t_1,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right) f\left(\underline{s}, \underline{X}_{t_2,T}\right)\right| \\
&\qquad + E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\left(f\left(\underline{s}, \underline{X}_{t_2,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)\right)\right| \\
&\leq C_f\, E\left|\left(f\left(\underline{s}, \underline{X}_{t_1,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right)\right| + C_f\, E\left|f\left(\underline{s}, \underline{X}_{t_2,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)\right| \\
&\leq \frac{C}{T}.
\end{aligned}$$

Now the sought-after result follows immediately. □

We stay in the product scenario, which acts as a guiding thread throughout this subsection. The last result concerns itself with the change of the companion process' argument.

Lemma A.10.
Suppose Assumptions 3 as well as Assumption 4 for $k = 1$ and Assumption 6 are fulfilled. Then, for all $h, k \in \{1, \ldots, T\}$ and $-TD_T \leq r, l \leq TD_T$ and $\underline{s} \in [-S, S]^d$ it holds

$$f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right) = f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right) + \mathcal{O}_P\left(D_T\right).$$

At this, the $\mathcal{O}_P$-term is independent of h, k, r, l and $\underline{s}$.

Proof. We want to benefit from the fact that while changing the argument but holding on to the index the difference between the two processes can be bounded nicely as it is stated in the second part of Lemma 2.3. Therefore, the first step is to create this kind of differences by

$$\begin{aligned} &E\left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right| \\ &\leq E\left|\left(f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right| \\ &\quad + E\left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right)\left(f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right)\right|. \end{aligned} \tag{A.179}$$

We see that the two summands of (A.179) are of the same building type. Hence, we limit ourselves to the examination of the first and transfer the results to the second one. Due to the boundedness of f, we can bound the second summand above by

$$C_f\, E\left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right|.$$

With use of the Lipschitz condition stated in Assumption 2, we obtain

$$\left\| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right\|_1 \leq C_{Lip}\left\| \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right) - \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right\|_1,$$

which, in turn, can be bounded by

$$C\left|\frac{h+r}{T} - \frac{h}{T}\right| \leq C\, D_T$$

using the second part of Lemma 2.3 and the fact that $|r| \leq TD_T$ holds. Consequently, we obtain

$$E\left|\left(f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right| = \mathcal{O}_P\left(D_T\right).$$

Since this results stays valid for the last summand of (A.179) as well, this lead to

$$f\left(\underline{s}, \widetilde{\underline{X}}_{h+r_1}\left(\frac{h+r_1}{T}\right)\right) f\left(\underline{s}, \widetilde{\underline{X}}_{k+l}\left(\frac{k+l}{T}\right)\right)$$
$$= f\left(\underline{s}, \widetilde{\underline{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \widetilde{\underline{X}}_{k+l}\left(\frac{k}{T}\right)\right) + \mathcal{O}_P\left(D_T\right),$$

which is the desired result. □

Remark A.11.
The unbounded counterpart of Lemma A.10, that is Lemma A.3, consists of two parts, whereas the lemma stated above only captures the product case. Since the proof of the first part of Lemma A.3 does not rely on the moments of f, the results stays true under Assumptions 4 for $k = 1, 3$ instead of Assumption 5.

At this point, we return to the demonstration of the results presented in Section 3.5. The next in line is Lemma 3.17, whose proof reads as follows:

Proof of Lemma 3.17. The two parts of the proof take the ones belonging to the proof of Lemma 3.7 as a guideline.

(i) Comparable to the aforementioned proof, we distinguish between three cases:

   a) $t \notin EP$,

   b) $t \in EP_1$ or

   c) $t_1 \in EP_2$.

   Again, we begin with the first one and have

$$E^\star f\left(\underline{s}, \underline{X}^\star_{t,T}\right) \leq \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| f\left(\underline{s}, \underline{X}_{t+r,T}\right)\right| \leq \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} C_f \leq C_f.$$

   Now we proceed with the endpoint cases. As before, we limit ourselves to the examination of case c). Then, it holds

$$\begin{aligned} E^\star f\left(\underline{s}, \underline{X}^\star_{t,T}\right) &\leq \frac{1}{2TD_T+1}\left(\sum_{r=-TD_T}^{T-t} \left| f\left(\underline{s}, \underline{X}_{t+r,T}\right)\right| + \sum_{r=T-t+1}^{TD_T} \left| f\left(\underline{s}, \underline{X}_{t-r,T}\right)\right|\right) \\ &\leq \frac{1}{2TD_T+1}\left(\sum_{r=-TD_T}^{T-t} C_f + \sum_{r=T-t+1}^{TD_T} C_f\right) \\ &= C_f, \end{aligned}$$

   which finishes the first part.

(ii) As in part (ii) of the proof belonging to Lemma 3.7, it suffices to look at the following cases:

a) $t_1, t_2 \notin EP$,

b) $t_1, t_2 \in EP_1$,

c) $t_1, t_2 \in EP_2$,

d) $t_1 \in EP_1$, $t_2 \notin EP$ or

e) $t_1 \in EP_2$, $t_2 \notin EP$,

starting with case a). From the proof of the second part of Lemma 3.7, we know already

$$
\begin{aligned}
&\mathrm{Cov}^\star\left(f\left(\underline{s}, \underline{X}^\star_{t_1,T}\right), f\left(\underline{s}, \underline{X}^\star_{t_2,T}\right)\right)\\
&= \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^\star_{t_2+r,T}\right)\\
&\qquad - \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^\star_{t_2+l,T}\right).
\end{aligned}
$$

Thus, taking the absolute value gives

$$
\begin{aligned}
&\left|\mathrm{Cov}^\star\left(f\left(\underline{s}, \underline{X}^\star_{t_1,T}\right), f\left(\underline{s}, \underline{X}^\star_{t_2,T}\right)\right)\right|\\
&\leq \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left|f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^\star_{t_2+r,T}\right)\right|\\
&\qquad + \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \left|f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) f\left(\underline{s}, \underline{X}^\star_{t_2+l,T}\right)\right|\\
&\leq \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} C_f^2 + \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} C_f^2\\
&=: C_{EP,1}
\end{aligned}
$$

for some positive constant $C_{EP,1} < \infty$. Moving on to the endpoint cases, we limit ourselves again to the case where $t_1, t_2 \in EP_2$ holds. As in the said proof, w.l.o.g. we suppose $t_1 \geq t_2$ and hence $T - t_2 \geq T - t_1$. This gives

$$
\begin{aligned}
&\left|\mathrm{Cov}^\star\left(f\left(\underline{s}, \underline{X}^\star_{t_1,T}\right), f\left(\underline{s}, \underline{X}^\star_{t_2,T}\right)\right)\right| \\
&= \left|\frac{1}{2\,TD_T+1}\sum_{r=-TD_T}^{T-t_1}\right. \\
&\Bigg(\Bigg(f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) - \frac{1}{2\,TD_T+1}\Bigg(\sum_{r=-TD_T}^{T-t_1} f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) + \sum_{r=T-t_1+1}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_1-r,T}\right)\Bigg)\Bigg) \\
&\cdot \Bigg(f\left(\underline{s}, \underline{X}^\star_{t_2+r,T}\right) - \frac{1}{2\,TD_T+1}\Bigg(\sum_{r=-TD_T}^{T-t_2} f\left(\underline{s}, \underline{X}^\star_{t_2+r,T}\right) + \sum_{r=T-t_2+1}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_2-r,T}\right)\Bigg)\Bigg)\Bigg) \\
&\qquad + \frac{1}{2\,TD_T+1}\sum_{r=T-t_1+1}^{T-t_2} \\
&\Bigg(\Bigg(f\left(\underline{s}, \underline{X}^\star_{t_1-r,T}\right) - \frac{1}{2\,TD_T+1}\Bigg(\sum_{r=-TD_T}^{T-t_1} f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) + \sum_{r=T-t_1+1}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_1-r,T}\right)\Bigg)\Bigg) \\
&\cdot \Bigg(f\left(\underline{s}, \underline{X}^\star_{t_2+r,T}\right) - \frac{1}{2\,TD_T+1}\Bigg(\sum_{r=-TD_T}^{T-t_2} f\left(\underline{s}, \underline{X}^\star_{t_2+r,T}\right) + \sum_{r=T-t_2+1}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_2-r,T}\right)\Bigg)\Bigg)\Bigg) \\
&\qquad + \frac{1}{2\,TD_T+1}\sum_{r=T-t_2+1}^{TD_T} \\
&\Bigg(\Bigg(f\left(\underline{s}, \underline{X}^\star_{t_1-r,T}\right) - \frac{1}{2\,TD_T+1}\Bigg(\sum_{r=-TD_T}^{T-t_1} f\left(\underline{s}, \underline{X}^\star_{t_1+r,T}\right) + \sum_{r=T-t_1+1}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_1-r,T}\right)\Bigg)\Bigg) \\
&\qquad \cdot \Bigg(f\left(\underline{s}, \underline{X}^\star_{t_2-r,T}\right) - \frac{1}{2\,TD_T+1}\Bigg(\sum_{r=-TD_T}^{T-t_2} f\left(\underline{s}, \underline{X}^\star_{t_2+r,T}\right) \\
&\qquad\qquad + \left.\sum_{r=T-t_2+1}^{TD_T} f\left(\underline{s}, \underline{X}^\star_{t_2-r,T}\right)\Bigg)\Bigg)\Bigg)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{T-t_1} \left(C_f + \frac{1}{2TD_T+1} \left(\sum_{r=-TD_T}^{T-t_1} C_f + \sum_{r=T-t_1+1}^{TD_T} C_f \right) \right) \\
&\qquad\qquad \cdot \left(C_f + \frac{1}{2TD_T+1} \left(\sum_{r=-TD_T}^{T-t_2} C_f + \sum_{r=T-t_2+1}^{TD_T} C_f \right) \right) \\
&\quad + \frac{1}{2TD_T+1} \sum_{r=T-t_1+1}^{T-t_2} \left(C_f + \frac{1}{2TD_T+1} \left(\sum_{r=-TD_T}^{T-t_1} C_f + \sum_{r=T-t_1+1}^{TD_T} C_f \right) \right) \\
&\qquad\qquad \cdot \left(C_f + \frac{1}{2TD_T+1} \left(\sum_{r=-TD_T}^{T-t_2} C_f + \sum_{r=T-t_2+1}^{TD_T} C_f \right) \right) \\
&\quad + \frac{1}{2TD_T+1} \sum_{r=T-t_2+1}^{TD_T} \left(C_f + \frac{1}{2TD_T+1} \left(\sum_{r=-TD_T}^{T-t_1} C_f + \sum_{r=T-t_1+1}^{TD_T} C_f \right) \right) \\
&\qquad\qquad\qquad\qquad \cdot \left(C_f + \frac{1}{2TD_T+1} \left(\sum_{r=-TD_T}^{T-t_2} C_f + \sum_{r=T-t_2+1}^{TD_T} C_f \right) \right) \\
&=: C_{EP,3}
\end{aligned}
$$

for some finite constant $C_{EP,3} > 0$ using Assumption 4. The amiss cases can be handled as described in the proof of the second part of Lemma 3.7 and lead to bounds specified by positive finite constants $C_{EP,2}, C_{EP,4}$ and $C_{EP,5}$, respectively. Hence, by defining

$$C_{EP,2} := \max_{i=1,\dots,5} C_{EP,i}$$

the second part and thus the whole proof is finished.

□

Now we focus on the convergence of the boostrap variance with the following proof:

Proof of Theorem 3.18. The proof orientates itself on the one of Theorem 3.8. Again, we assume T to be large enough such that the indices belonging to one endpoint group cannot be found in one single bootstrap block. Due to the independence of the blocks, we start by splitting up the bootstrap variance similar to the aforesaid proof, that is

$$
\begin{aligned}
&\operatorname{Var}^\star \left(\sum_{t=1}^{T} w_{t,T} \, f\left(\underline{s}, \underline{X}_{t,T}^\star\right) \right) \\
&= \operatorname{Var}^\star \left(\sum_{t=1}^{L_T \lceil (TD_T+1)/L_T \rceil} w_{t,T} \, f\left(\underline{s}, \underline{X}_{t,T}^\star\right) \right) + \operatorname{Var}^\star \left(\sum_{t=L_T \lceil (TD_T+1)/L_T \rceil + 1}^{L_T \lfloor (T-TD_T)/L_T \rfloor} w_{t,T} \, f\left(\underline{s}, \underline{X}_{t,T}^\star\right) \right) \\
&\qquad\qquad + \operatorname{Var}^\star \left(\sum_{t=L_T \lfloor (T-TD_T)/L_T \rfloor + 1}^{T} w_{t,T} \, f\left(\underline{s}, \underline{X}_{t,T}^\star\right) \right)
\end{aligned}
$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \tag{A.180}$$

This leads to an individual examination of the variance parts. Following the same argumentation as in the proof of Theorem 3.8, we obtain both $\mathrm{I} \le C_1\, d_T^{\frac{\delta}{2+\delta}}$ and $\mathrm{III} \le C_2\, d_T^{\frac{\delta}{2+\delta}}$. As this bounds rank among the class $o(1)$, we examine again only the first bootstrap variance term of (A.180) further. By reason of the independence of the bootstrap blocks, it holds again

$$\begin{aligned}
\mathrm{II} = &\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \\
&\qquad \cdot \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{tL_T+j+r,T}\right) f\left(\underline{s}, \underline{X}_{tL_T+l+r,T}\right) \right. \\
&\qquad \left. - \frac{1}{(2\,TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{tL_T+j+r,T}\right) \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{X}_{tL_T+l+k,T}\right) \right).
\end{aligned} \tag{A.181}$$

The next step will be the transformation of the bootstrap covariance into the one belonging to the real world. With use of Lemma A.9, we exchange the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ with the companion process and get

$$\begin{aligned}
&\sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \\
&\cdot \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\right) \right. \\
&\qquad\qquad + \mathcal{O}_P\left(T^{-1}\right) \\
&\qquad - \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \right. \\
&\qquad\qquad \left. \left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right) \right) + \mathcal{O}_P\left(T^{-1}\right) \right)
\end{aligned}$$

$$
\begin{aligned}
\leq & \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \\
& \cdot \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\right) \right. \\
& \qquad - \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \right. \\
& \qquad\qquad \left.\left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right)\right)\right) + \mathcal{O}_P\left(\frac{L_T}{T}\right)
\end{aligned}
\tag{A.182}
$$

instead of (A.181). Due to the fact that $\mathcal{O}_P\left(L_T/T\right)$ belongs to the class of $o_P(1)$ per definition, we leave the second summand of (A.182) behind and concentrate on the first. Before we can continue with the change of the argument of $\left(\underline{\widetilde{X}}_t(u)\right)$ in order to eliminate the dependence of summation index of the inner sum, we transform the sums as seen in the proof of Theorem 3.8. With h denoting the lag, the first summand of (A.182) can be rewritten as

$$
\begin{aligned}
& \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T} \\
& \qquad \cdot \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right) \right. \\
& \qquad\qquad \cdot f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \\
& \qquad - \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right) \right. \\
& \qquad\qquad \left.\left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j+k}{T}\right)\right)\right)\right).
\end{aligned}
\tag{A.183}
$$

Now Lemma A.10 is used to change the argument. Hence, we get in lieu of (A.183)

$$
\begin{aligned}
& \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T} \\
& \cdot \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\right) \right. \\
& \qquad - \frac{1}{(2\,TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right)\right.
\end{aligned}
$$

$$\cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\right)\Bigg)\Bigg) + \mathcal{O}_P\left(L_T\, D_T\right) \quad \text{(A.184)}$$

to work with. As the arguments are now independent of the summation indices, we use Lemma 3.6 to turn the bootstrap covariance into the real world one. Since it holds $\mathcal{O}_P\left(L_T\, D_T\right) \subseteq \mathcal{O}_P\left(L_T\, D_T^{\frac{3+\delta}{2(4+\delta)}}\right) \subseteq o_P(1)$ as seen in the proof of Theorem 3.8, our attentions lays again on the first summand of (A.184). Because we can copy exactly the lines of the aforementioned proof, we skip these steps here and move directly on with

$$\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right). \quad \text{(A.185)}$$

Next, we want the third sum of (A.185) to disappear. On account of this, we proceed analogously to the proof of Theorem 3.8 in terms of notation and approach. Nevertheless, due to a different summability condition we have to modify the last steps. Therefore, we start immediately with the examination of the difference between (A.185) and

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right). \quad \text{(A.186)}$$

In doing so, we obtain

$$\Bigg|\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)$$
$$- \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=1}^{L_T} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)\Bigg|$$
$$\le \frac{C_1}{L_T} \sum_{h=1}^{L_T-1} h \sum_{|k|\ge\lceil h/2\rceil} \frac{B}{l(k)}$$
$$\le \frac{C_1}{L_T} \sum_{h=1}^{L_T-1} h^{1-\tilde{\delta}} \sum_{|k|\ge\lceil h/2\rceil} k^{\tilde{\delta}} \frac{B}{l(k)}$$
$$\le C_1\, \frac{L_T^{1-\tilde{\delta}}}{L_T} \sum_{h=1}^{L_T-1} \sum_{|k|\ge\lceil h/2\rceil} k^{\tilde{\delta}} \frac{B}{l(k)}$$

$$\leq C_1 \, L_T^{-\tilde{\delta}} \sum_{k \in \mathbb{Z}} k^{1+\tilde{\delta}} \frac{B}{l(k)}$$
$$= \mathcal{O}\left(L_T^{-\tilde{\delta}}\right).$$

Because $\mathcal{O}\left(L_T^{-\tilde{\delta}}\right)$ belongs to the class $o(1)$, we are allowed to proceed with (A.186). Now the elimination of the lag out of the numerator in the argument of $\underline{\widetilde{X}}_h(\cdot)$ in equation (A.186) remains. Once again, we follow the lines of the proof of Lemma 2.13 the same way we did in the proof of Theorem 3.8 and bound the first summand of (A.186) by

$$\begin{aligned}
&\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} w_{t+h,T} \, w_{t,T} \\
&\qquad \cdot \left(\operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) + \mathcal{O}\left(\frac{|h|}{T}\right)\right) \\
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} w_{t+h,T} \, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \\
&\qquad + \mathcal{O}\left(\frac{L_T}{T}\right),
\end{aligned} \tag{A.187}$$

where L_T/T tends to 0 as T tends to infinity. Hence, we go on with the first term of (A.187). The last step is to show

$$\begin{aligned}
&\lim_{T\to\infty} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} w_{t+h,T} \, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \\
&\qquad = \sum_{h=-\infty}^{\infty} V_h\left(\underline{s}_1, \underline{s}_2\right)
\end{aligned}$$

with $V_h\left(\underline{s}_1, \underline{s}_2\right)$ as in Lemma 2.13. This happens to be done exactly as in the proof of Theorem 3.8 and is therefore omitted here. □

After having demonstrated all of the needed results to show the bounded version of the bootstrap CLT, this is the only remaining proof in this subsection. Therefore, we move directly on to said demonstration as it can be found below.

Proof of Theorem 3.19. Again, we assume that the number of indices belonging to one bootstrap block is strictly smaller than $d_T^{-\frac{\delta}{2(1+\delta)}}$. Then, we divide the sum in question similarly to the previous proof, that is

$$\begin{aligned}
&\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \\
&= \sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) + \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \\
&\qquad + \sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{A.188}$$

Using the same argumentation as in the proof of Theorem 3.9 in combination with part (i) of Lemma 3.17, we obtain $\mathrm{I} \leq C_1\, d_T^{\frac{\delta}{2(2+\delta)}}$ and $\mathrm{III} \leq C_2\, d_T^{\frac{\delta}{2(2+\delta)}}$. These two bounds pertain to the class $o(1)$ and are therefore negligible. Hence, we can move on to the second sum of equation (A.188). First, we rewrite the sum as seen in the above-mentioned proof leading to

$$\mathrm{II} = \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \sum_{j=1}^{L_T} w_{tL_T+j,T}\, \bar{f}^{\star}\left(\underline{s}, \underline{X}^{\star}_{tL_T+j,T}\right) =: \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \xi^{\star}_{t,T}.$$

Due to the independence of the bootstrap blocks, the $\left(\xi^{\star}_{t,T}\right)$ inherit the independence. This sets the course for the application of the classical central limit theorem based on Lyapunov's condition. From Theorem 3.8 we know

$$P - \lim_{T\to\infty} \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \mathrm{Var}^{\star}\left(\xi^{\star}_{t,T}\right) = \sigma^2_{as}. \tag{A.189}$$

The verification that $(\xi_{t,T})$ is equipped with finite absolute moments of order $2+\delta$ can be done rapidly by remembering that f is bounded. Consequently, we get

$$\begin{aligned}
E^{\star}\left|\xi^{\star}_{t,T}\right|^{2+\delta} &\leq \left(2\sum_{j=1}^{L_T} w_{tL_T+j,T}\, C_f\right)^{2+\delta} \\
&\leq C\left(\sum_{j=1}^{L_T} d_T^{1/2}\, \mathbb{1}_{\left\{w_{tL_T+j,T}>0\right\}} C_f\right)^{2+\delta} \\
&\leq C\, d_T^{\frac{2+\delta}{2}} \left(\sum_{j=1}^{L_T} \mathbb{1}_{\left\{w_{tL_T+j,T}>0\right\}}\right)^{2+\delta}.
\end{aligned} \tag{A.190}$$

At this point, we can move on to show the fulfillment of Lyapunov's condition. A combination of (A.189) and (A.190) with due regard to equation (A.111) established in the proof of Theorem 3.9 leads to

$$\frac{\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} E^\star \left|\xi^\star_{t,T}\right|^{2+\delta}}{\left(\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \mathrm{Var}^\star\left(\xi^\star_{t,T}\right)\right)^{\frac{2+\delta}{2}}} = \frac{C\, d_T^{-1} L_T^{1+\delta} d_T^{\frac{2+\delta}{2}}}{\left(\sigma^2\left(\underline{s},\underline{s}\right)\right)^{\frac{2+\delta}{2}} + o_P(1)} = \mathcal{O}_P\left(L_T^{1+\delta} d_T^{\delta/2}\right).$$

The consultation of Assumption 6 shows $L_T^{1+\delta} d_T^{\delta/2} = o(1)$. Hence, Lyapunov's condition is satisfied, and thus we have

$$\sup_{v\in\mathbb{R}} \left| P^\star\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}^\star\left(\underline{s}, \underline{X}^\star_{t,T}\right) \le v\right) - \Phi\left(\frac{v}{\sigma\left(\underline{s},\underline{s}\right)}\right)\right| \xrightarrow{P} 0$$

as $T \to \infty$ following the same argumentation as in the proof of Theorem 3.19. This terminates the proof. □

With the demonstration of the last theorem, we close the proving part belonging to Section 3.5.

A.2.4. Proofs of Section 3.6

This section deals with the proofs appurtenant to the findings stated in Section 3.6. We follow the stages yet seen in Subsection A.2.2, which contain additional results in form of determinations of suitable good sets.

However, the first result to deal with is addresses the sums of weighted covariance differences, to wit Lemma 3.21. The belonging proof reads as follows:

Proof of Lemma 3.21. The proof is geared to the one belonging to Lemma 3.11. We begin again by considering the absolute value of the double sum, which we bound by

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\right|. \tag{A.191}$$

Since the function f is bounded, we do not rely on the use of Hoelder's inequality in general in order to deal with the later occurring exponents as in the proof of Lemma 3.11. Instead, we only need the Cauchy-Schwarz inequality as a special case. First, we split the covariance up inserting the truncated version of the companion process with truncation parameter $M := \lceil |h|/2\rceil$ as seen in (2.13) as follows:

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right|\\
&=\left|\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right.\right.\\
&\qquad\qquad\qquad-\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right),\\
&\qquad\qquad\qquad\left.\left.f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right|\\
&\qquad\qquad+\left|\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right),\right.\right.\\
&\qquad\qquad\qquad f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\\
&\qquad\qquad\qquad\qquad\left.\left.-\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right)\right|\\
&=:\mathrm{I}+\mathrm{II}.
\end{aligned}\tag{A.192}$$

Following the argumentation in the proof of Lemma 3.11 at this point, we limit ourselves to the examination of term I and obtain again

$$\begin{aligned}
&E\left(\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right.\right.\\
&\qquad\qquad\left.-\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\qquad\qquad\left.\cdot\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\right)\\
&\qquad\qquad+E\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right.\\
&\qquad\qquad\left.-\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\qquad\qquad\qquad\cdot E\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\\
&=:\mathrm{Ia}+\mathrm{Ib}
\end{aligned}\tag{A.193}$$

to bound term I with. Now we use the Cauchy-Schwarz inequality for the first time and get for subterm Ia as upper bound

$$\begin{aligned}
&\left(E\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right.\right.\\
&\qquad\qquad\left.\left.-\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|^2\right)^{1/2}\\
&\qquad\qquad\qquad\qquad\cdot\left(E\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|^2\right)^{1/2}\\
&\le C\left(\left(E\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right.\right.\right.\\
&\qquad\qquad\left.\left.-\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\right)^{1/2}\\
&\qquad\qquad\qquad\qquad\left.\cdot\left(E\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\right)^{1/2}\right)\\
&=: C\cdot\mathrm{Iaa}\cdot\mathrm{Iab}
\end{aligned}$$

due to the boundedness of the function f. We continue again with the first factor. With help of the Lipschitz condition (2.11), subterm Iaa can be bounded by

$$C\left(E\left|\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)-\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right|\right)^{1/2}\le C\left(\sum_{|j|\ge M}\frac{B}{l(j)}\right)^{1/2}.$$

Using the Lipschitz condition of Assumption 2 anew, we establish an upper bound for subterm Iab as follows:

$$\mathrm{Iab}\le|\underline{s}_1-\underline{s}_2|_1^{1/2}\left(Eg\left(\underline{\widetilde{X}}_h\left(\frac{t}{T}\right),\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)^{1/2}.$$

Now we have a closer look at the expectation from above. It holds

$$\begin{aligned}
&Eg\left(\underline{\widetilde{X}}_h\left(\frac{t}{T}\right),\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\\
&\qquad\le E\left|g\left(\underline{\widetilde{X}}_h\left(\frac{t}{T}\right),\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-g\left(\underline{\widetilde{X}}_h(u),\underline{\widetilde{X}}_h(u)\right)\right|+Eg\left(\underline{\widetilde{X}}_1(u),\underline{\widetilde{X}}_1(u)\right)
\end{aligned}$$

for some $u\in[0,1]$ because of the stationarity of $g\left(\underline{\widetilde{X}}_t(u),\underline{\widetilde{X}}_t(u)\right)$ in u. Note that the last summand, $Eg\left(\underline{\widetilde{X}}_1(u),\underline{\widetilde{X}}_1(u)\right)$, does not depend on h or t. Besides, we have

$$E\left|g\left(\underline{\widetilde{X}}_h\left(\frac{t}{T}\right),\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-g\left(\underline{\widetilde{X}}_h(u),\underline{\widetilde{X}}_h(u)\right)\right|\le C_{g,2}$$

due to Assumption 8. Altogether, this means we can bound Iab via

$$\mathrm{Iab}\le C\,|\underline{s}_1-\underline{s}_2|_1^{1/2},$$

and thus it holds

$$\mathrm{Ia} \le C \left(\sum_{|j| \ge M} \frac{B}{l(j)} \right)^{1/2} |\underline{s}_1 - \underline{s}_2|_1^{1/2} .$$

Next, we concentrate on Ib. Having the fulfillment of the Lipschitz condition (2.11) in mind and the fact that it holds

$$E \left| f \left(\underline{s}_1, \underline{\widetilde{X}}_0 \left(\frac{t}{T} \right) \right) - f \left(\underline{s}_2, \underline{\widetilde{X}}_0 \left(\frac{t}{T} \right) \right) - \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)} \left(\frac{t}{T} \right) \right) - f \left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)} \left(\frac{t}{T} \right) \right) \right) \right| \le C \sum_{|j| \ge M} \frac{B}{l(j)},$$

as seen in the proof of Lemma 3.11, we get

$$\mathrm{Ib} \le C \sum_{|j| \ge M} \frac{B}{l(j)} |\underline{s}_1 - \underline{s}_2|_1 .$$

As a result, we have

$$\mathrm{I} \le C \left(\left(\sum_{|j| \ge M} \frac{B}{l(j)} \right)^{1/2} |\underline{s}_1 - \underline{s}_2|_1^{1/2} + \sum_{|j| \ge M} \frac{B}{l(j)} |\underline{s}_1 - \underline{s}_2|_1 \right) .$$

Following exactly the lines in the proof of Lemma 3.11, this bound can be transformed to get

$$\mathrm{I} \le C \left(\sum_{|j| \ge M} \frac{B}{l(j)} \right)^{1/2} |\underline{s}_1 - \underline{s}_2|_1^{1/2} .$$

Thus, as explained before, we obtain the same bound for II as well. In conclusion, we get

$$\left| \mathrm{Cov} \left(f \left(\underline{s}_1, \underline{\widetilde{X}}_0 \left(\frac{t}{T} \right) \right) - f \left(\underline{s}_2, \underline{\widetilde{X}}_0 \left(\frac{t}{T} \right) \right), f \left(\underline{s}_1, \underline{\widetilde{X}}_h \left(\frac{t}{T} \right) \right) - f \left(\underline{s}_2, \underline{\widetilde{X}}_h \left(\frac{t}{T} \right) \right) \right) \right| \le C \left(\sum_{|j| \ge M} \frac{B}{l(j)} \right)^{1/2} |\underline{s}_1 - \underline{s}_2|_1^{1/2} .$$

This means (A.191) can be bounded by

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T \lceil (TD_T+1)/L_T \rceil +1}^{L_T \lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \left(\sum_{|j| \ge M} \frac{B}{l(j)} \right)^{1/2} |\underline{s}_1 - \underline{s}_2|_1^{1/2} ,$$

and considering Assumption 3, we obtain

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$\leq \bar{C}_{DC}\, |\underline{s}_1 - \underline{s}_2|_1^{1/2}.$$

This finishes the proof. $\square$

At this point, we can identify our first sequence of good sets as done in the next lemma:

Lemma A.12.
Assuming Assumption 3 plus Assumptions 4 for $k = 1$ and 6 are fulfilled. Then, there exist subsets $\left(\bar{B}_T\right)_{T\in\mathbb{N}}$ of Ω with $P\left(\bar{B}_T\right) \to 1$ as $T \to \infty$ such that

$$\operatorname{Var}^\star\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}^\star_{t,T}\right) - f\left(\underline{s}_2, \underline{X}^\star_{t,T}\right)\right)\right)$$
$$= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$+ \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right)$$

holds on $\bar{B}_T$ for any $\underline{s}_1, \underline{s}_2 \in [-S, S]^d$. Here, neither $\underline{s}_1$ nor $\underline{s}_2$ has any influence on said $\mathcal{O}$-terms.

Proof. During this proof, we will follow the lines of the proof of Lemma A.6, which is this lemma's counterpart for unbounded functions f. Again, we assume T to be large enough to ensure $L_T < d_T^{-\frac{\delta}{2(1+\delta)}}$. That being said, we start by splitting the bootstrap variance similar to the aforementioned proof. In doing so, we get

$$\begin{aligned}
&\mathrm{Var}^\star\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right)\\
&=\mathrm{Var}^\star\left(\sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\left(f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right)\\
&\qquad+\mathrm{Var}^\star\left(\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right)\\
&\qquad\qquad+\mathrm{Var}^\star\left(\sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right)\\
&=:\bar{R}^I_T+\bar{V}_T+\bar{R}^{II}_T. \qquad\qquad (A.194)
\end{aligned}$$

With the use of the same arguments as in the proof of Lemma A.6 and the second part of Lemma 3.17, we obtain $\bar{R}^I_T=\mathcal{O}\left(d_T^{\frac{\delta}{2+\delta}}\right)$ and $\bar{R}^{II}_T=\mathcal{O}\left(d_T^{\frac{\delta}{2+\delta}}\right)$. In contrast to said proof, we have already $\mathcal{O}$-terms, which means no adjustment is needed. Therefore, we go on with the remaining term of (A.194), that is V_T. We start by transforming the variance of sums into sums of covariances, that is

$$\begin{aligned}
V_T&=\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1}\sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\,w_{tL_T+l,T}\\
&\qquad\mathrm{Cov}^\star\left(f\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right),f\left(\underline{s}_1,\underline{X}^\star_{tL_T+l,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{tL_T+l,T}\right)\right)\\
&=\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1}\sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\,w_{tL_T+l,T}\,\mathrm{Cov}^\star\left(f\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right),f\left(\underline{s}_1,\underline{X}^\star_{tL_T+l,T}\right)\right)\\
&\quad-\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1}\sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\,w_{tL_T+l,T}\,\mathrm{Cov}^\star\left(f\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right),f\left(\underline{s}_2,\underline{X}^\star_{tL_T+l,T}\right)\right)\\
&\quad-\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1}\sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\,w_{tL_T+l,T}\,\mathrm{Cov}^\star\left(f\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right),f\left(\underline{s}_1,\underline{X}^\star_{tL_T+l,T}\right)\right)\\
&\quad+\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1}\sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\,w_{tL_T+l,T}\,\mathrm{Cov}^\star\left(f\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right),f\left(\underline{s}_2,\underline{X}^\star_{tL_T+l,T}\right)\right).
\end{aligned}$$
$$(A.195)$$

Now we remember again the steps made in the proof of Theorem 3.18 to turn equation (A.195) into

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$- \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$- \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$+ \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\ \mathrm{Cov}\left(f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$+ \bar{R}_T^{III} + \bar{R}_T^{IV} + \bar{R}_T^{V} + \mathcal{O}\left(L_T^{-1}\right) + \mathcal{O}\left(\frac{L_T}{T}\right)$$
$$= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$+ \bar{R}_T^{III} + \bar{R}_T^{IV} + \bar{R}_T^{V} + \mathcal{O}\left(L_T^{-1}\right) + \mathcal{O}\left(\frac{L_T}{T}\right) \tag{A.196}$$

with rest terms $\bar{R}_T^{III}, \bar{R}_T^{IV}$ and $\bar{R}_T^{V}$. As before, these terms consist each of four subparts originating from the non-combined covariances. Consequently, we have $\bar{R}_T^{III} := \sum_{i_1,i_2=1}^{2} \bar{R}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)$, $\bar{R}_T^{IV} := \sum_{i_1,i_2=1}^{2} \bar{R}_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)$ and $\bar{R}_T^{V} := \sum_{i_1,i_2=1}^{2} \bar{R}_T^{V}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)$, where the indices i and j indicate the used combination of $\underline{s}_1$ and $\underline{s}_2$. The combined $\mathcal{O}_P$-rates are $\bar{R}_T^{III} = \mathcal{O}_P\left(\frac{L_T}{T}\right)$, $\bar{R}_T^{IV} = \mathcal{O}_P\left(L_T\, D_T\right)$ and $\bar{R}_T^{V} = \mathcal{O}_P\left(\frac{L_T}{TD_T}\right)$, respectively. Now we examine the twelve individual rest terms further starting with the them belonging to the first rest term in (A.196). We have for every combination of $i_1, i_2 \in \{1,2\}$

$$\bar{R}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = \Bigg| \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}$$
$$\cdot \left(\left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{X}_{tL_T+j+r,T}\right) f\left(\underline{s}_{i_2}, \underline{X}_{tL_T+l+r,T}\right) \right.\right.$$
$$\left. - \frac{1}{\left(2\,TD_T+1\right)^2} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{X}_{tL_T+j+r,T}\right) \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{X}_{tL_T+l+k,T}\right) \right)$$
$$- \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right)$$

$$\cdot f\left(\underline{s}_{i_2}, \widetilde{X}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\right)$$
$$-\frac{1}{\left(2TD_T+1\right)^2}\left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \widetilde{X}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right.$$
$$\left.\left.\left.\cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \widetilde{X}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right)\right)\right)\right|.$$

Thus, set

$$\bar{B}_T^I\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) := \left\{\omega \in \Omega \,\middle|\, \bar{R}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq \frac{L_T}{TD_T}\right\} = \left\{\omega \in \Omega \,\middle|\, \frac{TD_T}{L_T}\, \bar{R}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq 1\right\}.$$

Following the scheme in the counterpart proof for unbounded f, we need to verify that $P\left(\bar{B}_T^I\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right)$ tends to 1 as T tends to ∞. We have

$$P\left(\bar{B}_T^I\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) \geq 1 - E\left(\frac{TD_T}{L_T}\, \bar{R}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right).$$

The expectation above can be bounded by

$$E\left(\frac{TD_T}{L_T}\, \bar{R}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) = \mathcal{O}\left(\frac{TD_T}{L_T}\cdot\frac{L_T}{T}\right) = \mathcal{O}\left(D_T\right),$$

which leads to $\lim_{T\to\infty} \bar{B}_T^I\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = 1$ for all $i_1, i_2 \in \{1,2\}$. A suitable combination of all subsets based on all possible pairs of indices is given by

$$\bar{B}_T^I := \bigcap_{i,j\in\{1,2\}} \bar{B}_T^I\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right).$$

Due to the way $\bar{B}_T^I$ is formed, we have $\lim_{T\to\infty} \bar{B}_T^I = 1$ as well. Next, we look at the second rest term of equation (A.196), namely $\bar{R}_T^{II}$, and its subterms. It holds for all $i_1, i_2 \in \{1,2\}$

$$\bar{R}_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = \left|\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \sum_{j=1}^{L_T} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{l=\max\{1,1-h\}}^{\min\{L_T, L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}\right.$$
$$\cdot\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \widetilde{X}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right)\right.$$
$$\cdot f\left(\underline{s}_{i_2}, \widetilde{X}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)$$
$$-\frac{1}{\left(2TD_T+1\right)^2}\left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \widetilde{X}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right)\right.$$
$$\left.\left.\cdot\sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \widetilde{X}_{tL_T+j+k}\left(\frac{tL_T+j+k}{T}\right)\right)\right)\right)$$

$$
\begin{aligned}
&- \left(\frac{1}{2\,TD_T + 1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T + j + h}{T} \right) \right) \right. \\
&\qquad \cdot f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T + j}{T} \right) \right) \\
&- \frac{1}{(2\,TD_T + 1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T + j + h}{T} \right) \right) \right. \\
&\qquad \left. \left. \left. \cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T + j}{T} \right) \right) \right) \right) \right| .
\end{aligned}
$$

Thus, we define

$$
\begin{aligned}
\bar{B}_T^{II}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) &:= \left\{ \omega \in \Omega \,\middle|\, \bar{R}_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq L_T\, D_T^{\frac{1+\delta}{2}} \right\} \\
&= \left\{ \omega \in \Omega \,\middle|\, \left(L_T\, D_T^{\frac{1+\delta}{2}} \right)^{-1} \bar{R}_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq 1 \right\}.
\end{aligned}
$$

Similar to before, we have

$$
P\left(\bar{B}_T^{II}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) \geq 1 - E\left(\left(L_T\, D_T^{\frac{1+\delta}{2}} \right)^{-1} \bar{R}_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \right).
$$

Focusing on the expectation, we get

$$
E\left(\left(L_T\, D_T^{\frac{1+\delta}{2}} \right)^{-1} \bar{R}_T^{IV}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \right) = \mathcal{O}\left(L_T^{-1} D_T^{-\frac{1+\delta}{2}} L_T\, D_T \right) = \mathcal{O}\left(D_T^{\frac{1-\delta}{2}} \right),
$$

which implies $\lim_{T\to\infty} \bar{B}_T^{II}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = 1$ for every pair (i, j) with $i_1, i_2 \in \{1, 2\}$. Again, we compose the intersection

$$
\bar{B}_T^{II} := \bigcap_{i,j\in\{1,2\}} \bar{B}_T^{II}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)
$$

to obtain the sought-after convergence $\lim_{T\to\infty} \bar{B}_T^{II} = 1$. Lastly, it holds

$$
\begin{aligned}
\bar{R}_T^{V}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = &\left| \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor - 1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{l=\max\{1,1-h\}}^{\min\{L_T, L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T} \right. \\
&\cdot \left(\frac{1}{2\,TD_T + 1} \sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T + j + h}{T} \right) \right) \right. \\
&\qquad \cdot f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T + j}{T} \right) \right) \\
&- \frac{1}{(2\,TD_T + 1)^2} \left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T + j + h}{T} \right) \right) \right.
\end{aligned}
$$

$$\cdot \sum_{k=-TD_T}^{TD_T} f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\right)\Bigg)$$

$$- \operatorname{Cov}\left(f\left(\underline{s}_{i_1}, \underline{\widetilde{X}}_{tL_T+j+h}\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}_{i_2}, \underline{\widetilde{X}}_{tL_T+j}\left(\frac{tL_T+j}{T}\right)\right)\right)\Bigg)\Bigg|$$

for all $i_1, i_2 \in \{1,2\}$. A suitable subset of Ω is

$$\begin{aligned}\bar{B}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) &:= \left\{\omega \in \Omega \,\middle|\, \bar{R}_T^V\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right\} \\ &= \left\{\omega \in \Omega \,\middle|\, \frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\, \bar{R}_T^V\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) \leq 1\right\}.\end{aligned}$$

This can be seen, once again, by checking the convergence of the corresponding probability. Because of

$$P\left(\bar{B}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) \geq 1 - E\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\, \bar{R}_T^V\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right),$$

we look at

$$E\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\, \bar{R}_T^V\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)\right) = \mathcal{O}\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T} \cdot \frac{L_T}{TD_T}\right) = \mathcal{O}\left((TD_T)^{-\delta/4}\right).$$

This gives the desired result, to wit $\lim_{T\to\infty} \bar{B}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right) = 1$ for all $i_1, i_2 \in \{1,2\}$. The intersection

$$\bar{B}_T^{III} := \bigcap_{i,j \in \{1,2\}} \bar{B}_T^{III}\left(\underline{s}_{i_1}, \underline{s}_{i_2}\right)$$

inherits the convergences, and we have $\lim_{T\to\infty} \bar{B}_T^{III} = 1$. Recapitulatory, we combine the previously defined subsets again by intersection and get

$$\bar{B}_T := \bar{B}_T^{I} \cap \bar{B}_T^{II} \cap \bar{B}_T^{III} \tag{A.197}$$

with

$$P\left(\bar{B}_T\right) = P\left(\bar{B}_T^{I} \cap \bar{B}_T^{II} \cap \bar{B}_T^{III}\right) \longrightarrow 1$$

as $T \to \infty$. As before the remaining $\mathcal{O}$-terms of (A.196), which are valid on Ω, retain their validity on $\bar{B}_T$ leaving us with

$$\begin{aligned}
&\operatorname{Var}^\star\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}^\star_{t,T}\right) - f\left(\underline{s}_2, \underline{X}^\star_{t,T}\right)\right)\right) \\
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T} \\
&\quad \cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) \\
&\qquad + \mathcal{O}\left(d_T^{\frac{\delta}{2+\delta}}\right) + \mathcal{O}\left(\frac{L_T}{TD_T}\right) + \mathcal{O}\left(L_T\, D_T^{\frac{1+\delta}{2}}\right) + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right) + \mathcal{O}\left(\frac{L_T}{T}\right)
\end{aligned} \tag{A.198}$$

for every $\omega \in \bar{B}_T$. At this point, we have a closer look at the $\mathcal{O}$-terms in order to identify those which can be included into others. We begin with the comparison of the first and second-to-last term of (A.198). From the proof of Lemma A.6, we know

$$\mathcal{O}\left(d_T^{\frac{\delta}{2(1+\delta)}}\right) \subseteq \mathcal{O}\left(L_T^{-1}\right)$$

already. Since it holds

$$d_T^{\frac{\delta}{2+\delta}} < d_T^{\frac{\delta}{2(1+\delta)}}$$

due to the larger denominator of the exponent on the right-hand side, we can conclude

$$\mathcal{O}\left(d_T^{\frac{\delta}{2+\delta}}\right) \subseteq \mathcal{O}\left(L_T^{-1}\right).$$

Again from said proof, we know

$$\mathcal{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right) \subseteq \mathcal{O}\left(L_T^{-1}\right).$$

Because of

$$\frac{L_T}{T} \leq \left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}},$$

we can deduce for the last term of equation (A.198)

$$\mathcal{O}\left(\frac{L_T}{T}\right) \subseteq \mathcal{O}\left(L_T^{-1}\right).$$

Moving on to the third and second-to-last term, we consider Assumption 6 to get

$$
\begin{aligned}
L_T\, D_T^{\frac{1+\delta}{2}} &< d_T^{-\frac{\delta}{2(1+\delta)}} \left(d_T^{-\frac{1}{2+\delta}} T^{-1} \right)^{\frac{1+\delta}{2}} \\
&\leq d_T^{-\frac{\delta}{2(1+\delta)}} \left(d_T^{-\frac{1}{2+\delta}+1} \right)^{\frac{1+\delta}{2}} \\
&= d_T^{\frac{1+\delta+2\delta^2+\delta^3}{2(1+\delta)(2+\delta)}}
\end{aligned}
$$

leading to the verification of

$$
d_T^{\frac{\delta}{2(1+\delta)}} > d_T^{\frac{1+\delta+2\delta^2+\delta^3}{2(1+\delta)(2+\delta)}}
$$

since it holds $L_T^{-1} > d_T^{\frac{\delta}{2(1+\delta)}}$ if T is sufficiently large. Because the base is the same, we relocate the inequality to the exponents and check

$$
\frac{1+\delta+2\delta^2+\delta^3}{2(1+\delta)(2+\delta)} > \frac{\delta}{2(1+\delta)}
$$

instead. This is equivalent to $1-\delta+\delta^2+\delta^3 > 0$, which is fulfilled for every $\delta \in (0,1)$. Thus, it follows

$$
\mathcal{O}\left(L_T\, D_T^{\frac{1+\delta}{2}} \right) \subseteq \mathcal{O}\left(L_T^{-1} \right).
$$

Lastly, we look at the second and fourth term of (A.198). As it holds

$$
TD_T > (TD_T)^{\frac{4-\delta}{4}},
$$

we get

$$
\mathcal{O}\left(\frac{L_T}{TD_T} \right) \subseteq \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}} \right).
$$

Having all that in mind, we can state for every $\omega \in \bar{B}_T$

$$
\begin{aligned}
&\mathrm{Var}^\star \left(\sum_{t=1}^{L_T\lfloor T/L_T \rfloor} w_{t,T} \left(f\left(\underline{s}_1, \underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{t,T}^\star\right) \right) \right) \\
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil+1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \\
&\quad \cdot \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) \right) \\
&\qquad + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}} \right) + \mathcal{O}\left(L_T^{-1} \right),
\end{aligned}
$$

and the proof is completed. $\square$

The next sequence of good sets whose existence we want to show deals with the function g in place of f. This is a novelty compared to the already determined sets both in this section and in its unbounded counterpart.

Lemma A.13.
Under Assumptions 6 and 8, there exist subsets $\left(\bar{K}_T\right)_{T\in\mathbb{N}}$ of Ω with $P\left(\bar{K}_T\right) \to 1$ as $T \to \infty$, on which it holds

$$\sum_{t=0}^{\lfloor T/L_T\rfloor-1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}} E^\star\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right)^{\frac{2+\delta}{2}} \leq L_T^{1+\delta} \tag{A.199}$$

for the prevailing T.

Proof. This proof models itself on the proof of Lemma A.7. To put in briefly, we will establish an upper bound for the expectation of the right-hand side of equation (A.199). Hereinafter, this upper bound will be relaxed to identify a suitable subset, on which the bound circumvents the mere validity in probability. Therefore, we start by considering the expectation of the expression in question, that is

$$E\left(\sum_{t=0}^{\lfloor T/L_T\rfloor-1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}} E^\star\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right)^{\frac{2+\delta}{2}}\right).$$

Since the second factor is the only stochastic term, we draw the expectation into the product and obtain

$$\sum_{t=0}^{\lfloor T/L_T\rfloor-1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}} E\left(E^\star\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right)^{\frac{2+\delta}{2}}\right) \tag{A.200}$$

Before we continue with the investigation of (A.200) as a whole, we focus on the bootstrap expectation. It holds

$$\begin{aligned} &E^\star\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right)^{\frac{2+\delta}{2}} \\ &= \left\|\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}^{\frac{2+\delta}{2}} \\ &\leq \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\|g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\right)^{\frac{2+\delta}{2}}. \end{aligned}$$

Inserting this bound into (A.200) gives

$$\begin{aligned}
&\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}} E\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\left\|g\left(\underline{X}^{\star}_{tL_T+j,T},\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\right)^{\frac{2+\delta}{2}}\\
&=\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}}\left\|\sum_{j=1}^{L_T} w_{tL_T+j,T}\left\|g\left(\underline{X}^{\star}_{tL_T+j,T},\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\right\|_{\frac{2+\delta}{2}}^{\frac{2+\delta}{2}}\\
&\leq\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\left\|\left\|g\left(\underline{X}^{\star}_{tL_T+j,T},\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\right\|_{\frac{2+\delta}{2}}\right)^{\frac{2+\delta}{2}}
\end{aligned}\tag{A.201}$$

to go on with. Now we turn our attention again to one part of the expression, namely the nested $\mathcal{L}^{\frac{2+\delta}{2}}$-norm of g. Due to slight differences between Assumptions 7 and 8, we cannot follow the lines in the proof belonging to Lemma A.7 exactly. Nevertheless, the basic procedure is the same. First, we focus on the inner norm in (A.201). We want to benefit from the stationarity of the companion process in combination with the function g. Thus, there is the need of transforming the arguments of g. In order to do so, we a get by adding self-canceling differences

$$\begin{aligned}
&\left\|g\left(\underline{X}^{\star}_{tL_T+j,T},\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\\
&=\left\|g\left(\underline{X}^{\star}_{tL_T+j,T},\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\\
&-\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right),\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}\\
&+\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right),\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}\\
&\qquad-\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u),\underline{\widetilde{X}}_{tL_T+j+r}(u)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}\\
&\qquad\qquad+\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u),\underline{\widetilde{X}}_{tL_T+j+r}(u)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}.
\end{aligned}\tag{A.202}$$

Note that we followed the restriction of only investigating non-endpoint cases as explained yet in the proof of Lemma A.7. Incorporating (A.202) into the outer norm in (A.201) leads to

$$
\begin{aligned}
&\left\| \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star} \right\|_{\frac{2+\delta}{2}} \\
&\leq \left\| \left\| g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star} \right. \\
&\qquad - \left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right), \right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left.\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} \right\|_{\frac{2+\delta}{2}} \\
&\qquad + \left\| \left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right), \right.\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} \\
&\qquad - \left.\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u), \underline{\widetilde{X}}_{tL_T+j+r}(u)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} \right\|_{\frac{2+\delta}{2}} \\
&\qquad + \left\| \left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u), \underline{\widetilde{X}}_{tL_T+j+r}(u)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} \right\|_{\frac{2+\delta}{2}} \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad\qquad (\mathrm{A}.203)
\end{aligned}
$$

In the following, we investigate these terms one by one and start with I. Said term can be bounded by $\frac{C_{g,1}}{T}$ similarly to the respective term in the aforementioned proof but with help of part (ii) of Assumption 8. Thus, we move on to the second summand of equation (A.203), we get

$$
\begin{aligned}
\mathrm{II} \leq &\left(E\left| \left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right), \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. -g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u), \underline{\widetilde{X}}_{tL_T+j+r}(u)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} \right|^{\frac{2+\delta}{2}} \right)^{\frac{2}{2+\delta}} \\
= &\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} E\left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right), \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \right.\right. \\
&\qquad \left.\left. -g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u), \underline{\widetilde{X}}_{tL_T+j+r}(u)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} \left\| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right), \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) \right. \right. \\
&\qquad\qquad \left. \left. - g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u), \underline{\widetilde{X}}_{tL_T+j+r}(u)\right) \right\|_{\frac{2+\delta}{2}}^{\frac{2+\delta}{2}} \right)^{\frac{2}{2+\delta}} \\
&\leq \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} C_2 \right)^{\frac{2}{2+\delta}} \\
&\leq C_1. \qquad\qquad \text{(A.204)}
\end{aligned}
$$

Now only term III of (A.203) is left to be examined. It holds

$$
\begin{aligned}
\text{Ic} &= \left(E\left(\left(\left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u), \underline{\widetilde{X}}_{tL_T+j+r}(u)\right) \right|^{\frac{2+\delta}{2}} \right)^{\frac{2}{2+\delta}} \right)^{\frac{2+\delta}{2}} \right) \right)^{\frac{2}{2+\delta}} \\
&= \left(E\left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}(u), \underline{\widetilde{X}}_{tL_T+j+r}(u)\right) \right|^{\frac{2+\delta}{2}} \right) \right)^{\frac{2}{2+\delta}} \\
&= \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} E\left| g\left(\underline{\widetilde{X}}_{1}(u), \underline{\widetilde{X}}_{1}(u)\right) \right|^{\frac{2+\delta}{2}} \right)^{\frac{2}{2+\delta}} \\
&\leq \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} C_2 \right)^{\frac{2}{2+\delta}} \\
&\leq C_2 \qquad\qquad \text{(A.205)}
\end{aligned}
$$

because of the finite absolute moments of order $2+\delta$ the function g is equippend with in combination with the stationarity of the companion process. Taken all together, equations (A.204) and (A.205) as well as the upper bound for term I of (A.203) yield

$$
\begin{aligned}
&C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{\frac{2-\delta}{2}} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left(T^{-1}+1\right) \right)^{\frac{2+\delta}{2}} \\
&\leq C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{2} \\
&\leq C\, \frac{d_T^{-1}}{L_T} L_T^2\, d_T \\
&= C\, L_T
\end{aligned}
$$

as an upper bound for equation (A.201). Thereby, we have shown

$$
\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{\frac{2-\delta}{2}} E^{\star} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T}\right) \right)^{\frac{2+\delta}{2}} = \mathcal{O}_P\left(L_T\right).
$$

Hence, we take

$$P\left(\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}} E^\star\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right)^{\frac{2+\delta}{2}} > L_T\right)$$

into consideration. The next step is to alter the lower bound above to the extend that this probability is part of the class $o(1)$. To do so, we work with the following set

$$\begin{aligned}\bar{K}_T := \Bigg\{\omega\in\Omega\,\Bigg|\, &\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}}\\ &\cdot E^\star\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right)^{\frac{2+\delta}{2}} \le L_T^{1+\delta}\Bigg\}.\end{aligned}$$

At this point, we use the technique seen in the proof of Lemma A.7, among others, again and consider the corresponding expectation. Hence, by making use of the previously established result we obtain

$$\begin{aligned}&E\left(L_T^{-(1+\delta)}\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}} E^\star\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^\star_{tL_T+j,T}, \underline{X}^\star_{tL_T+j,T}\right)\right)^{\frac{2+\delta}{2}}\right)\\ &= \mathcal{O}\left(L_T^{-(1+\delta)}L_T\right)\\ &= \mathcal{O}\left(L_T^{-\delta}\right),\end{aligned}$$

which tends to 0 while T tends to ∞. Therefore, it holds $\lim_{T\to\infty} P\left(\bar{K}_T\right) = 1$, and the proof is completed.

□

At this point, all needed good set are established and we can return to Section 3.6 to benefit from them. Thus, we continue with the proof of Lemma 3.22.

Proof of Lemma 3.22. First, we define again subsets $\left(\bar{\Omega}_T\right)_{T\in\mathbb{N}}$ of Ω. This time, we will use the subsets $\left(\bar{B}_T\right)_{T\in\mathbb{N}}$ and $\left(\bar{K}_T\right)_{T\in\mathbb{N}}$ having their seeds in Lemma A.12 and Lemma A.13, respectively. We set

$$\bar{\Omega}_T := \bar{B}_T \cap \bar{K}_T.$$

Then, we obtain $\lim_{T\to\infty} P\left(\bar{\Omega}_T\right) = 1$, and this T-depending subset is the foundation on which our proof relies.

As seen before in the two proofs belonging to tightness results, we follow the idea of Arcones and Yu (1994). This version combines the steps of the proof of version (b) of

Lemma 2.18 with the bootstrap techniques used in the one appertaining to Lemma 3.12. Thus, we start again by dividing the sum in equation (3.6) to get one sum containing the indices of whole independent blocks and a second comprising the remaining indices. This leads to

$$\begin{aligned}
&P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\lambda\right)\\
&\leq P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\frac{\lambda}{2}\right)\\
&\quad+P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\frac{\lambda}{2}\right). \qquad \text{(A.206)}
\end{aligned}$$

With use of Markov's inequality, the second sum of (A.206) can be bounded via

$$\begin{aligned}
&P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\frac{\lambda}{2}\right)\\
&\leq \frac{2}{\lambda}\,E^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|\right)\\
&\leq \frac{8}{\lambda}\,C_f\,L_T\,d_T^{1/2}\\
&= o\left(d_T^{-\frac{\delta}{2(1+\delta)}+\frac{1}{2}}\right)\\
&= o\left(d_T^{\frac{1}{2(1+\delta)}}\right),
\end{aligned}$$

which tends to 0 as T tends to ∞. Thus, as in the proof of the counterpart lemma with f unbounded, that is Lemma 3.12, we are left with the examination of the first sum of (A.141). In the following, we will use the same notation as in the proof of the said lemma. Therefore, we omit the repetition at this point and go on with rewriting the remaining sum of (A.141) identically to before. Hence, we have

$$P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right|>\frac{\lambda}{2}\right)$$

$$\le P^\star\left(2\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\nu^\star_T\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{6}\right)$$

$$+P^\star\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r}}\left|\nu^\star_T\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{6}\right)$$

$$+P^\star\left(2\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3\,r_k}}\left|\nu^\star_T\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{6}\right)$$

$$=:\mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{A.207}$$

The next step we are aiming to perform is again the application of Bernstein's inequality to the terms II and III. Therefore, considering the inner sum of

$$\nu^\star_T\left(\underline{s}_1,\underline{s}_2\right)=\sum_{t=1}^{L_T\lfloor T/L_T\rfloor-1}\sum_{j=1}^{L_T} w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right)\right)$$

and, this time, using the bound C_f of the function f, we get

$$\begin{aligned}\left|\sum_{j=1}^{L_T} w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right)\right)\right| &\le 4\,L_T\,d_T^{1/2}C_f\\ &< C\,d_T^{-\frac{\delta}{2(1+\delta)}+\frac{1}{2}}\\ &= C\,d_T^{\frac{1}{2(1+\delta)}}\end{aligned} \tag{A.208}$$

as an upper bound for $T>\bar{T}$ comparable to the proof of Lemma 3.12. The second requirement for the use of Bernstein's inequality is to find an upper bound for the bootstrap variance of $\nu^\star_T\left(\underline{s}_1,\underline{s}_2\right)$. By Lemma 3.21 and Lemma A.12, we have

$$\begin{aligned}&\mathrm{Var}^\star\left(\nu^\star_T\left(\underline{s}_1,\underline{s}_2\right)\right)\\ &=\mathrm{Var}^\star\left(\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\sum_{j=1}^{L_T} w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{tL_T+j,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{tL_T+j,T}\right)\right)\right)\\ &=\mathrm{Var}^\star\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-f\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T} \\
&\quad \cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) \\
&\qquad\qquad + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right) \\
&\leq \bar{C}_{DC} \left|\underline{s}_1 - \underline{s}_2\right|_1^{1/2} + \mathcal{O}\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right) + \mathcal{O}\left(L_T^{-1}\right). \qquad \text{(A.209)}
\end{aligned}
$$

As seen in the proof of Lemma 3.12, our aim is to rewrite the bounds established in (A.208) and (A.209) in such a way that they are dependent on r_k. As we can choose the bounds of r_{k_T} to our liking, we aim to set the bounds mentioned earlier in relation to r_{k_T}. Later on, we will replace r_{k_T} by r_k for k presC ribed by the outer sum in term II. To combine all ratios occurring in the equations (A.208) and (A.209), we consider

$$
\max\left\{ \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}, L_T^{-1}, d_T^{\frac{1}{2(1+\delta)}} \right\}. \qquad \text{(A.210)}
$$

In the following, we compare the ratios written above in order to minimize the number of arguments over which we maximize. As the comparison between the ratios originating from Lemma A.12 took already place in the respective proof, we only need to compare the ratios of (A.208) with $d_T^{\frac{1}{2(1+\delta)}}$. Starting with L_T^{-1}, it holds

$$
L_T^{-1} > d_T^{\frac{\delta}{2(1+\delta)}} > d_T^{\frac{1}{2(1+\delta)}}
$$

for $T > \bar{T}$ due to Assumption 6. Thus, we work with

$$
\max\left\{ \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}, L_T^{-1} \right\} \qquad \text{(A.211)}
$$

for $T > \bar{T}$ instead of (A.210). Next, we propose a lower and and an upper bound for r_{k_T}. Afterwards, we will, comparable to the proceeding in the proof appurtenant to Lemma 3.12, show that our choice for these bounds goes along with the requirements imposed on r_{k_T}. Hence, for $T > \bar{T}$ we state

$$
\left(\max\left\{ \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}, L_T^{-1} \right\}\right)^{\frac{4(1+\delta)(2+\delta)}{\delta^4(1-\delta)}} \leq r_{k_T} \leq L_T^{-\frac{2(1+\delta)}{\delta^2}}. \qquad \text{(A.212)}
$$

The first thing to verify is the existence of such a r_{k_T} with $k_T \in \mathbb{N}$ given these bounds. To do so, we show that there is an actual gap between the lower and the upper bound. This has to be done individually for both arguments of the maximum. We begin with the verification of

$$
\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right)^{\frac{4(1+\delta)(2+\delta)}{\delta^4(1-\delta)}} < L_T^{-\frac{2(1+\delta)}{\delta^2}}. \qquad \text{(A.213)}
$$

This inequality is equivalent to

$$\left(\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right)^{\frac{2(2+\delta)}{\delta^2(1-\delta)}} < L_T^{-1},$$

which, in turn, can be rewritten as

$$L_T^{\frac{2(2+\delta)+\delta^2(1-\delta)}{\delta^2(1-\delta)}} < (TD_T)^{\frac{(2+\delta)(4-\delta)}{2\delta^2(1-\delta)}}. \tag{A.214}$$

Following Assumption 6, it holds

$$L_T^{\frac{2(2+\delta)+\delta^2(1-\delta)}{\delta^2(1-\delta)}} < d_T^{-\frac{\delta}{2(1+\delta)}\cdot\frac{2(2+\delta)+\delta^2(1-\delta)}{\delta^2(1-\delta)}} = d_T^{-\frac{2(2+\delta)+\delta^2(1-\delta)}{2\delta(1-\delta)(1+\delta)}}$$

for $T > \bar{T}$ and

$$(TD_T)^{\frac{(2+\delta)(4-\delta)}{2\delta^2(1-\delta)}} \geq d_T^{-\frac{\delta}{2+\delta}\cdot\frac{(2+\delta)(4-\delta)}{2\delta^2(1-\delta)}} = d_T^{-\frac{4-\delta}{2\delta(1-\delta)}}.$$

Therefore, we can check

$$d_T^{-\frac{4-\delta}{2\delta(1-\delta)}} > d_T^{-\frac{2(2+\delta)+\delta^2(1-\delta)}{2\delta(1-\delta)(1+\delta)}} \tag{A.215}$$

instead of (A.214). Due to the same base on both sides in (A.215), we narrow the comparison down to the exponents. Hence, we aim to verify

$$\frac{4-\delta}{2\delta(1-\delta)} > \frac{2(2+\delta)+\delta^2(1-\delta)}{2\delta(1-\delta)(1+\delta)},$$

which can be cut short to to $\delta - 2\delta^2 + \delta^3 > 0$. The validity of the latter inequality for every choice of $\delta \in (0,1)$ can easily be seen. Hence, this directs to the validity of equation (A.214) under the assumption of T being larger than $\bar{T}$ as well. Lastly, we are going to verify

$$\left(L_T^{-1}\right)^{\frac{4(1+\delta)(2+\delta)}{\delta^4(1-\delta)}} < L_T^{-\frac{2(1+\delta)}{\delta^2}}, \tag{A.216}$$

which is equivalent to

$$L_T^{-\frac{2(2+\delta)}{\delta^2(1-\delta)}} < L_T^{-1}.$$

Thus, it has to hold

$$\frac{2(2+\delta)}{\delta^2(1-\delta)} > 1,$$

which is fulfilled for our choice of δ since the numerator on the left-hand side is bigger than 1, whereas the denominator is smaller than 1. Therefore, (A.216) is valid as well. This proves the existence of r_{k_T} for $T > T_2$ under our pre-made assumptions. The importance of the upper bound in equation (A.212) becomes visible in the last part of this proof, namely during the examination of term I of equation (A.207). Until now, we ignored the difference between $\underline{s}_1$ and $\underline{s}_2$ in (A.209). But, again, owing to our set-up in which the calculations take place, to wit it holds $\rho\left(\underline{s}_1, \underline{s}_2\right) \le 3\, r_k$ with k determined by the outer sum of term II, we obtain automatically a bound of the desired form. Consequently, we have

$$\operatorname{Var}^\star\left(\nu_T^\star\left(\underline{s}_1, \underline{s}_2\right)\right) \le C\left(r_k^{1/2} + r_{k_T}^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}}\right) \le C\left(r_k^{1/2} + r_k^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}}\right) \tag{A.217}$$

for $T > \bar{T}$ due to $r_{k_T} < r_k$ for all $k = 0, 1, \ldots, r_{k_T}$. As in the proof of Lemma 3.12, we want to concentrate the sum in (A.217) in order to bound the bootstrap-variance with a single rate. For r small enough, both r_k for all $k = 0, 1, \ldots, r_{k_T}$ and r_{k_T} are smaller than 1. Hence, we compare the exponents to find the smaller and therefore dominating one. Supposing the smaller exponent is $\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}$, we check the following relation:

$$\frac{1}{2} - \frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)} > 0. \tag{A.218}$$

An equivalent expression reads as $2(1+\delta)(2+\delta) > \delta^4(1-\delta)$. This inequality is fulfilled for every $\delta \in (0,1)$, as then, the right-hand side is smaller than 1. In consequence, equation (A.218) is true and $\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}$ dominates 1/2. Hence, for $T > \bar{T}$ and r sufficiently small we consider

$$\operatorname{Var}^\star\left(\nu_T^\star\left(\underline{s}_1, \underline{s}_2\right)\right) \le C\, r_k^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} \tag{A.219}$$

in lieu of (A.217). After the establishment of suitable bounds to use in the application of Bernstein's inequality, we bring now λ into focus. Similarly to the corresponding proof for unbounded functions f, we define a sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that

$$2\sum_{k\in\mathbb{N}} \lambda_k \le \frac{\lambda}{6}$$

holds true for r adequately small. When it comes to the more precise definition of the used λ_k for $k = 1, \ldots, k_T$, we turn away from the aforementioned proof but presume

$$\lambda_k := r_k^{\frac{\delta^4(1-\delta)}{16(1+\delta)(2+\delta)}} \vee \left(\frac{4}{\bar{C}} \log\left(D(k)\right) r_k^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}}\right)^{1/2}$$

instead. Once again, $D(k) = \mathcal{O}\left(r_k^{-d}\right)$ guarantees that $(\lambda_k)_{k=1}^{k_T}$ is able to be totaled for $T \to \infty$. Now we have made all necessary preparations to be able to apply Bernstein's inequality to term II. In doing so, we obtain

$$\begin{aligned}
&\limsup_{T\to\infty} P^\star\left(2\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathcal{F}_k, \underline{s}_2\in\mathcal{F}_{k-1} \\ \rho(\underline{s}_1,\underline{s}_2)\le 3\, r_k}} \left|\nu_T^\star\left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{6}\right) \\
&\le \limsup_{T\to\infty} P^\star\left(2\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathcal{F}_k, \underline{s}_2\in\mathcal{F}_{k-1} \\ \rho(\underline{s}_1,\underline{s}_2)\le 3\, r_k}} \left|\nu_T^\star\left(\underline{s}_1, \underline{s}_2\right)\right| > 2\sum_{k=1}^{k_T} \lambda_k\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{T\to\infty} \sum_{k=1}^{k_T} P^\star \left(\sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r_k}} \left|\nu_T^\star\left(\underline{s}_1, \underline{s}_2\right)\right| > \lambda_k \right) \\
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} D(k) D(k-1) \exp\left(-\frac{1}{2} \cdot \frac{\lambda_k^2}{C\, r_k^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} + 4/3C\, d_T^{\frac{1}{2(1+\delta)}} \lambda_k} \right) \\
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} \exp\left(2\log(D(k)) - \frac{1}{2} \cdot \frac{\lambda_k^2}{C\, r_k^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} + C\, r_{k_T}^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} \lambda} \right) \\
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} \exp\left(2\log(D(k)) - \bar{C} \frac{\lambda_k^2}{r_k^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}}} \right) \\
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2} \cdot \frac{\lambda_k^2}{r_k^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}}} \right) \\
&\leq \limsup_{T\to\infty} 2 \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2} r_k^{\frac{2\delta^4(1-\delta)}{16(1+\delta)(2+\delta)} - \frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} \right) \\
&\leq 2 \limsup_{T\to\infty} \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2}\, r_k^{-\frac{\delta^4(1-\delta)}{8(1+\delta)(2+\delta)}} \right) \\
&\leq 2 \sum_{k\in\mathbb{N}} \exp\left(-C\, r^{-\frac{\delta^4(1-\delta)}{8(1+\delta)(2+\delta)}} \right) \\
&\underset{r\to 0}{\longrightarrow} 0.
\end{aligned}
$$

Following the argumentation in the proof of Lemma 3.12, we move on to term III of (A.207) and get

$$
\begin{aligned}
&\limsup_{T\to\infty} P^\star \left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} \left|\nu_T^\star\left(\underline{s}_1, \underline{s}_2\right)\right| > \frac{\lambda}{6} \right) \\
&\leq \limsup_{T\to\infty} 2\, D^2(0) \exp\left(-\frac{1}{2} \cdot \frac{(\lambda/6)^2}{C\, r^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} + 4/3C\, d_T^{\frac{1}{2(1+\delta)}} \lambda/6} \right) \\
&\leq 2 \left(D\left(r, [-S,S]^d, \rho\right)\right)^2 \exp\left(-\frac{1}{2} \cdot \frac{1}{C\, r^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} + C\, r^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}}} \right) \\
&\leq 2 \left(\frac{2\, Sd}{r} + 1 \right)^{2d} \exp\left(-\frac{1}{2} \cdot \frac{1}{C\, r^{\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}}} \right) \\
&\leq \mathcal{O}\left(r^{-2d}\right) \exp\left(-\bar{C}\, r^{-\frac{\delta^4(1-\delta)}{4(1+\delta)(2+\delta)}} \right) \\
&\underset{r\to 0}{\longrightarrow} 0.
\end{aligned}
$$

As term I of equation (A.207) is not akin to the second and third one, we chance the approach as already seen in the proof of Lemma (3.12). This implies the use of a symmetrization lemma. First, note again that term I is equal to

$$P^{\star}\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\nu_T^{\star}\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{12}\right).$$

Let again $L_{t,T}^{\star}\left(\underline{s}\right)$ and $L_{t,T}^{\star,0}\left(\underline{s}\right)$ be defined as follows:

$$L_{t,T}^{\star}\left(\underline{s}\right):=\sum_{j=1}^{L_T} w_{tL_T+j,T}\, f\left(\underline{s},\underline{X}_{tL_T+j,T}^{\star}\right)$$

and

$$L_{t,T}^{\star,0}\left(\underline{s}\right):=\zeta_t\, L_{t,T}^{\star}\left(\underline{s}\right)$$

with $(\zeta_t)_{t=0}^{\lfloor T/L_T\rfloor-1}$ i.i.d. Rademacher variables independent to $(k_t)_{t=0}^{\lfloor T/L_T\rfloor-1}$ originating from Algorithm 3.1. For the same reason as in the proof of this lemma's version for unbounded functions, it is enough to look at

$$E^{\star}\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\nu_T^{\star}\left(\underline{s}_1,\underline{s}_2\right)\right|\right),$$

which, again, can be bounded by

$$2\,E^{\star}\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(L_{t,T}^{\star,0}\left(\underline{s}_1\right)-L_{t,T}^{\star,0}\left(\underline{s}_2\right)\right)\right|\right). \tag{A.220}$$

Likewise to the aforesaid proof, Hoeffding's inequality gives us

$$P^{\star}\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=0}^{\lfloor T/L_T\rfloor-1} L_{t,T}^{\star,0}\left(\underline{s}_1\right)-L_{t,T}^{\star,0}\left(\underline{s}_2\right)\right|>\hat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right)\eta\,\middle|\, L_{1,T}^{\star},\ldots,L_{L_T\lfloor T/L_T\rfloor,T}^{\star}\right)\le 2\exp\left(-\frac{\eta^2}{2}\right)$$

using

$$\hat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right):=\left(\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(L_{t,T}^{\star}\left(\underline{s}_1\right)-L_{t,T}^{\star}\left(\underline{s}_2\right)\right)^2\right)^{1/2} \tag{A.221}$$

defined for $\underline{s}_1, \underline{s}_2 \in [-S,S]^d$. This shows again that $\sum_{t=0}^{\lfloor T/L_T \rfloor -1} L_{t,T}^{\star,0}$ comes with sub-Gaussian increments conditionally on $L_{0,T}^\star, \ldots, L_{\lfloor T/L_T \rfloor -1,T}^\star$. Now it is again time to leave the path following exactly the lines of the proof of Lemma (3.12). For this, we use a different way to bound the difference in (A.221), that is

$$\left(L_{t,T}^\star(\underline{s}_1) - L_{t,T}^\star(\underline{s}_2)\right)^2 \leq 2^{1-\delta} \left|L_{t,T}^\star\right|_\infty^{\frac{2-\delta}{2}} \left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}} \rho(\underline{s}_1, \underline{s}_2)^{\frac{2+\delta}{2}} \tag{A.222}$$

for $\underline{s}_1, \underline{s}_2 \in [-S,S]^d$ as seen comparably in (A.171). In the following, we use the proof of Lemma 3.12 as a guideline but combine it with the calculations used in the proof of part (b) of Lemma 2.18. So the next step is again to establish a semimetric which suits us more. To this end, let Q_T be defined as

$$Q_T := 2^{\frac{2-\delta}{4}} \left(\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left|L_{t,T}^\star\right|_\infty^{\frac{2-\delta}{2}} \left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}} \right)^{1/2}. \tag{A.223}$$

Now the examination of $\left|L_{t,T}^\star\right|_\infty$ and $\left|L_{t,T}^\star\right|_{\mathrm{Lip}}$ falls into line. It holds

$$\left|L_{t,T}^\star\right|_\infty = \sup_{\underline{s}\in[-S,S]^d} \left| \sum_{j=1}^{L_T} w_{tL_T+j,T}\, f\left(\underline{s}, \underline{X}_{tL_T+j,T}^\star\right) \right| \leq C_f \sum_{j=1}^{L_T} w_{tL_T+j,T} \tag{A.224}$$

and

$$\begin{aligned} \left|L_{t,T}^\star\right|_{\mathrm{Lip}} &= \sup_{\substack{\underline{s}_1 \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \neq 0}} \frac{\left| \sum_{j=1}^{L_T} w_{tL_T+j,T} \left(f\left(\underline{s}_1, \underline{X}_{tL_T+j,T}^\star\right) - f\left(\underline{s}_2, \underline{X}_{tL_T+j,T}^\star\right)\right)\right|}{\rho(\underline{s}_1, \underline{s}_2)} \\ &\leq \sup_{\substack{\underline{s}_1 \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \neq 0}} \frac{\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}_{tL_T+j,T}^\star, \underline{X}_{tL_T+j,T}^\star\right) \rho(\underline{s}_1, \underline{s}_2)}{\rho(\underline{s}_1, \underline{s}_2)} \\ &= \sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}_{tL_T+j,T}^\star, \underline{X}_{tL_T+j,T}^\star\right) \end{aligned} \tag{A.225}$$

due to the Lipschitz condition (2.11). We do not use the bound stated in Assumption 3 concerning the weights because later on, we will make use of the fact that the number of weights unequal to 0 can be smaller than T. Combining (A.223), (A.224) and (A.225), we obtain for Q_T

$$\begin{aligned} Q_T \leq 2^{\frac{2-\delta}{4}} \Bigg(\sum_{t=0}^{\lfloor T/L_T \rfloor -1} & \left(C_f \sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{\frac{2-\delta}{2}} \\ & \cdot \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}_{tL_T+j,T}^\star, \underline{X}_{tL_T+j,T}^\star\right) \right)^{\frac{2+\delta}{2}} \Bigg)^{1/2}. \end{aligned} \tag{A.226}$$

At this point, we define the previously mentioned semimetric, namely

$$\hat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) \leq Q_T \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{2+\delta}{4}} =: \breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right).$$

Comparably to the proof of Lemma 3.12, this is in fact a semimetric because of

$$\breve{\rho}_T\left(\underline{s}_1, \underline{s}_2\right) \leq Q_T\left(\rho\left(\underline{s}_1, \underline{s}_3\right) + \rho\left(\underline{s}_3, \underline{s}_2\right)\right)^{\frac{2+\delta}{4}} \leq \breve{\rho}_T\left(\underline{s}_1, \underline{s}_3\right) + \breve{\rho}_T\left(\underline{s}_3, \underline{s}_2\right).$$

Returning to equation A.220, we look at at conditional expectation

$$E^\star\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(L_{t,T}^{\star,0}\left(\underline{s}_1\right) - L_{t,T}^{\star,0}\left(\underline{s}_2\right)\right) \right| \,\middle|\, L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star\right).$$

With the use of the maximal inequality for sub-Gaussian processes in Corollary 2.2.8 of van der Vaart and Wellner (2000), we obtain

$$\begin{aligned}
&E^\star\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(L_{t,T}^{\star,0}\left(\underline{s}_1\right) - L_{t,T}^{\star,0}\left(\underline{s}_2\right)\right) \right| \,\middle|\, L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star\right) \\
&= E^\star\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \breve{\rho}(\underline{s}_1, \underline{s}_2) \leq Q_T r_{k_T}^{\frac{2+\delta}{4}}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(L_{t,T}^{\star,0}\left(\underline{s}_1\right) - L_{t,T}^{\star,0}\left(\underline{s}_2\right)\right) \right| \,\middle|\, L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star\right) \\
&\leq C\left(\int_0^{Q_T r_{k_T}^{\frac{2+\delta}{4}}} \left(\log D\left(u, [-S,S]^d, \breve{\rho}_T\right)\right)^{1/2} du\right).
\end{aligned}$$

As before, it holds for the packing number from above

$$D\left(u, [-S,S]^d, \breve{\rho}_T\right) = D\left(\left(\frac{u}{Q_T}\right)^{\frac{4}{2+\delta}}, [-S,S]^d, \rho\right) \leq \left(\frac{2\,Sd}{\left(\frac{u}{Q_T}\right)^{\frac{4}{2+\delta}}} + 1\right)^d.$$

In order to obtain an integrable function, we replace the packing number by its upper bound. This leads to

$$\begin{aligned}
&E^\star\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(L_{t,T}^{\star,0}\left(\underline{s}_1\right) - L_{t,T}^{\star,0}\left(\underline{s}_2\right)\right) \right|\right) \\
&\leq C E^\star\left(\int_0^{Q_T r_{k_T}^{\frac{2+\delta}{4}}} \left(\log\left(\frac{2\,Sd}{\left(\frac{u}{Q_T}\right)^{\frac{4}{2+\delta}}} + 1\right)^d\right)^{1/2} du\right)
\end{aligned}$$

$$= C E^{\star} Q_T \int_0^{r_{k_T}^{\frac{2+\delta}{4}}} \left(\log\left(\frac{2\,Sd}{u^{\frac{4}{2+\delta}}} + 1 \right)^d \right)^{1/2} du. \tag{A.227}$$

Next, investigate the expectation and the integral separately. Starting with the first, we remember equation (A.226) and get

$$E^{\star} Q_T \leq C_1 \left(\sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \right)^{\frac{2-\delta}{2}} \right.$$
$$\left. \cdot E^{\star} \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^{\star}_{tL_T+j,T}, \underline{X}^{\star}_{tL_T+j,T} \right) \right)^{\frac{2+\delta}{2}} \right)^{1/2}. \tag{A.228}$$

With the help of Lemma A.13, we can work with

$$E^{\star} Q_T \leq C_1\, L_T^{\frac{1+\delta}{2}} \tag{A.229}$$

instead of (A.228). Now we focus on the integral and have

$$\int_0^{r_{k_T}^{\frac{2+\delta}{4}}} \left(\log\left(\frac{2\,Sd}{u^{\frac{4}{2+\delta}}} + 1 \right)^d \right)^{1/2} du \leq \int_0^{r_{k_T}^{\frac{2+\delta}{4}}} \left(\log\left(\frac{2\,Sd}{u^{\frac{4}{2+\delta}}} + 1 \right)^d \right)^{1/2} du$$

since it holds $\log(x+1) \leq x$ for $x > 0$. Going on, we use the upper bound proposed in (A.212) and get

$$\begin{aligned}
\int_0^{r_{k_T}^{\frac{2+\delta}{4}}} \left(\log\left(\frac{2\,Sd}{u^{\frac{4}{2+\delta}}} + 1 \right)^d \right)^{1/2} du &\leq C_2 \left[u^{\frac{\delta}{2+\delta}} \right]_0^{r_{k_T}^{\frac{2+\delta}{4}}} \\
&= C_2\, r_{k_T}^{\delta/4} \\
&\leq C_2\, L_T^{-\frac{2(1+\delta)}{\delta^2} \cdot \frac{\delta}{4}} \\
&= C_2\, L_T^{-\frac{1+\delta}{2\delta}}.
\end{aligned} \tag{A.230}$$

At this point, we bring the equations (A.227), (A.229) and (A.230) together and obtain

$$\begin{aligned}
&\limsup_{T\to\infty} E^{\star} \left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} \left| \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \left(L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \right) \right| \right) \\
&\leq \limsup_{T\to\infty} C\, L_T^{\frac{1+\delta}{2}}\, L_T^{-\frac{1+\delta}{2\delta}} \\
&= \limsup_{T\to\infty} C\, L_T^{-\frac{1-\delta^2}{2\delta}} \\
&= 0,
\end{aligned}$$

which closes the poof. □

As already explained several times before, the statement of the tightness result whose proof is written above combined with the bootstrap CLT in Section 3.5 allows for the proof of the bounded version of the bootstrap FCLT, which happens next.

Proof of Theorem 3.23. The proof can be executed exactly as its counterpart dealing with unbounded functions f. The only difference is the use of Theorem 3.19 and Lemma 3.22 instead of Theorem 3.9 and Lemma 3.12, respectively. □

The last demonstration concludes the proofs of the general bootstrap results.

A.3. Proofs Belonging to Chapter 4

This section occupies itself with the proofs of the results dedicated to render the simulation study in Section 4.2 possible. Because this study uses the special case dealing with ECFs, the findings which need to be proven address this framework as well. Since all of the results in Chapter 4 can be found in Section 4.1, there is no further subsectioning here.

To begin with, we have focus on the kernel function in the following proof:

Proof of Lemma 4.2. First of all, note that for T large enough it holds both

$$-\frac{u}{b_T} \leq -1 \quad \text{and} \quad \frac{1-u}{b_T} \geq 1. \tag{A.231}$$

Thus, we start by defining $z := \frac{y-u}{b_T}$. Then, classical substitution leads to

$$\frac{1}{b_T}\int_0^1 K\left(\frac{y-u}{b_T}\right) dy = \int_{-\frac{u}{b_T}}^{\frac{1-u}{b_T}} K(z)\, dz = \int_{-1}^{1} K(z)\, dz = 1$$

under the use of equation (A.231) and the validity of Assumption 9 for T sufficiently large. Now we are able to bound the difference between the weighted sum of the kernel function with different arguments and 1 the following way:

$$\begin{aligned}\left|\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) - 1\right| &= \left|\frac{1}{b_T}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}\left(K\left(\frac{t/T-u}{b_T}\right) - K\left(\frac{y-u}{b_T}\right)\right) dy\right| \\ &\leq \frac{1}{b_T}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}\left|K\left(\frac{t/T-u}{b_T}\right) - K\left(\frac{y-u}{b_T}\right)\right| dy \end{aligned}\tag{A.232}$$

again for suitable T. Next, we focus on the kernel difference and remember the Lipschitz continuity of the kernel function stated in Assumption 9. This allows us to bound said difference via

$$\left|K\left(\frac{t/T-u}{b_T}\right) - K\left(\frac{y-u}{b_T}\right)\right| \leq C_{Lip,K}\left|\frac{t/T-y}{b_T}\right| \leq \frac{C_{Lip,K}}{b_T T}.$$

This transforms the integral in (A.232) into

$$\int_{\frac{t-1}{T}}^{\frac{t}{T}} \frac{C_{Lip,K}}{b_T T}\, dy = \frac{C_{Lip,K}}{b_T T^2}.$$

The number of non-zero summands in (A.232) is of order $\mathcal{O}\left(b_T T\right)$ as seen in part (ii) of Remark 4.1. Having the prefactor $\left(b_T\right)^{-1}$ in mind, this leads, in consequence, to

$$\left| \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) - 1 \right| = \mathcal{O}\left(\frac{1}{b_T T}\right).$$

Thereupon, the proof is finished. □

The next proof belongs to a result, which deals with the difference between CF and EC, to wit Lemma 4.3. In comparison to the version in Jentsch et al. (2020), which can be found between the lines, we give an explicit proof concentrating on the needed steps to show this result, which eases the proof.

Proof of Lemma 4.3. Since the difference between the real and the imaginary part is the use of the sine in place of the cosine function in the latter, we restrict ourselves to the proof belonging to the real part. Afterwards, the imaginary counterpart can be deduced straight-away. First of all, we look at the absolute value of the difference in question and obtain

$$\begin{aligned} &\left|\Re\left(E\,\widehat{\varphi}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right)\right| \\ &\quad = \left| \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) E\cos\left(\langle \underline{s}, \underline{X}_{t,T}\rangle\right) - E\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_1(u)\rangle\right)\right|. \end{aligned} \tag{A.233}$$

Next, we want to benefit from the closeness between the local stationary process and its companion. Before we are able to do so, we need to incorporate the subtrahend in (A.233) into the sum. Stationarity of the companion process helps us at it and we get

$$\begin{aligned} &\left| \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(E\cos\left(\langle \underline{s}, \underline{X}_{t,T}\rangle\right) - E\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t(u)\rangle\right)\right)\right| \\ &\qquad + \left| \left(\frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) - 1\right) E\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_1(u)\rangle\right)\right| \\ &=: \mathrm{I} + \mathrm{II} \end{aligned} \tag{A.234}$$

as an upper bound to equation (A.233). Now we need to insert the companion process with t/T as argument, which yields

$$\begin{aligned} \mathrm{I} \leq &\frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) E\left|\cos\left(\langle \underline{s}, \underline{X}_{t,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\rangle\right)\right| \\ &+ \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) E\left|\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t(u)\rangle\right)\right| \end{aligned}$$

$$=: \mathrm{Ia} + \mathrm{Ib}. \tag{A.235}$$

We proceed in two stages. First, we look at the two differences above. After having found upper borders for them, we take sum and kernel into account. Starting with the difference in Ia, we obtain using Lemma 2.6

$$\left\| \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right) \rangle \right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t(u) \rangle \right) \right\|_1 \leq \frac{C_{B'}}{T}.$$

Moving on to the second difference, we utilise the Lipschitz condition (2.11) and the second part of Lemma 2.3 to get

$$\begin{aligned}
\left\| \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right) \rangle \right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t(u) \rangle \right) \right\|_1 &\leq C_{Lip} \left\| \underline{\widetilde{X}}_t\left(\frac{t}{T}\right) - \underline{\widetilde{X}}_t(u) \right\|_1 \\
&\leq C \left| \frac{t}{T} - u \right| \\
&\leq C\, b_T.
\end{aligned}$$

The last step is possible since the kernel K is always zero for arguments fulfilling $\left|\frac{t}{T} - u\right| > b_T$, and a non-zero kernel is the reason for the necessity of our calculations. At this point, we bring the kernel back into play. As already seen in the second part of Remark 4.1, the kernel is positive only for at most $2\lfloor b_T T \rfloor + 1$ indices of the sum. Since the established upper bounds for the differences are independent of the summation index, the number ob non-zero weights provided by the kernel can be set off against the prefactor leading to $\mathcal{O}(1)$-terms. Consequently, the combination with the upper bounds lead to $\mathrm{Ia} = \mathcal{O}\left(T^{-1}\right)$ and $\mathrm{Ib} = \mathcal{O}\left(b_T\right)$, respectively. From Assumption 9, we know that b_T dominates T^{-1} for T large enough. Therefore, we have

$$\mathcal{O}\left(T^{-1}\right) \subseteq \mathcal{O}\left(b_T\right).$$

Returning to equation (A.234), this leads to $\mathrm{I} = \mathcal{O}\left(b_T\right)$. Now we move on to the second summand said equation. Because the cosine is bounded by 1, only the difference is of interest. Consulting Lemma 4.2, we get $\mathrm{II} = \mathcal{O}\left(\left(b_T T\right)^{-1}\right)$. Note that both the rates of terms I and II are independent of $\underline{s}$. Thus, the supremum becomes obsolete in the upper bounds. The last step is now to reintegrate the prefactor. In doing so, we obtain

$$\begin{aligned}
&\sup_{\underline{s} \in [-S,S]^d} \left(b_T T\right)^{1/2} \Re\left(E\,\widehat{\varphi}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right) \\
&= \mathcal{O}\left(\left(b_T^3 T\right)^{1/2}\right) + \mathcal{O}\left(\left(b_T T\right)^{-1/2}\right) \\
&= o(1)
\end{aligned}$$

due to Assumption 9 and $b_T^3 T = o(1)$. As explained in the beginning, this result can be transferred directly to the imaginary part. Hence, the proof is finished. □

Using the lemma proven above, we are now able to demonstrate the validity of Theorem 4.4 as follows:

Proof of Theorem 4.4. Following the same argumentation as in the previous lemma, we confine ourselves to the proof of the first result. To start, we have a look at Theorem 3.19. The result stated there can be rewritten to become

$$\sup_{v\in\mathbb{R}}\left|P^\star\left((b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X^\star\left(u;\underline{s}\right)-E^\star\widehat{\varphi}_X^\star\left(u;\underline{s}\right)\right)\leq v\right)-\Phi\left(\frac{v}{\sigma\left(\underline{s},\underline{s}\right)}\right)\right|\xrightarrow{P}0 \tag{A.236}$$

as T to ∞ adapted to our setting. We know from Theorem 2.15 that the Gaussian distribution is the limit distribution of $(b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-E\,\widehat{\varphi}_X\left(u;\underline{s}\right)\right)$ for T tending to ∞. Comparing this expression with the sought-after result, we see a difference which appears once more in the subtrahend. Therefore, the first step will be to show that $(b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)$ has the same limit distribution as the difference above. At this point, Lemma 4.3 comes in handy. Taking said lemma into account, we obtain

$$\begin{aligned}
&\left|(b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)-(b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-E\,\widehat{\varphi}_X\left(u;\underline{s}\right)\right)\right|\\
&=\left|(b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)\right|\\
&=o_P(1).
\end{aligned}$$

This implies that the Gaussian distribution in equation (A.236) is also the limit of $(b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-E\,\widehat{\varphi}_X\left(u;\underline{s}\right)\right)$ for $T\to\infty$. Next, we consider the expression in question again. It holds

$$\begin{aligned}
&\sup_{v\in\mathbb{R}}\Bigg|P^\star\left((b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X^\star\left(u;\underline{s}\right)-E^\star\widehat{\varphi}_X^\star\left(u;\underline{s}\right)\right)\leq v\right)\\
&\qquad\qquad-P\left((b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)\leq v\right)\Bigg|\\
&\leq\sup_{v\in\mathbb{R}}\left|P^\star\left((b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X^\star\left(u;\underline{s}\right)-E^\star\widehat{\varphi}_X^\star\left(u;\underline{s}\right)\right)\leq v\right)-\Phi\left(\frac{v}{\sigma\left(\underline{s},\underline{s}\right)}\right)\right|\\
&\qquad+\sup_{v\in\mathbb{R}}\left|P\left((b_TT)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)\leq v\right)-\Phi\left(\frac{v}{\sigma\left(\underline{s},\underline{s}\right)}\right)\right|\\
&=o_P(1)+o(1)\\
&=o_P(1).
\end{aligned}$$

This finishes the proof. □

The last proof brings this short section to completion.

A.4. Proofs Belonging to Chapter 5

In this section, we find a combination of proofs dealing with a modification of the generalized theory established in Chapter 2 and with the special case addressing ECFs as introduced in Chapter 4. The reason behind that is the aim to remain as general as possible when it comes to new findings. Only if the need for these results is implied by the especial application of the basic theory to the ECF-case, we tailor said results directly to fit the situation.

A.4.1. Proofs of Section 5.2

The first subsection covers the proof of consistency result presented in Section 5.2 and the demonstration of the modified findings to incorporate $\varphi_X\left(u;\underline{s}\right)$ as well as some preparatory results.

At first, we aim to show a modified version of the bounded part of Theorem 2.19. Thus, we start with the proof of the altered covariance result in Lemma 5.2.

Proof of Lemma 5.2. The first equation to be shown deals with the real part, whereas the second one occupies itself with the imaginary part. Thus, the differences lies in the used trigonometric function. Therefore, it suffice to focus on one of them, and the other follows immediately.

According to version (b) of Lemma 2.13, it holds

$$\begin{aligned}
&\lim_{T\to\infty}\operatorname{Cov}\left(\Re\left(\left(b_T T\right)^{1/2}\widehat{\varphi}_X\left(u;\underline{s}_1\right)\right),\Re\left(\left(b_T T\right)^{1/2}\widehat{\varphi}_X\left(u;\underline{s}_2\right)\right)\right)\\
&=\sum_{h\in\mathbb{Z}}\lim_{T\to\infty}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)K\left(\frac{(t+h)/T-u}{b_T}\right)\\
&\qquad\cdot\operatorname{Cov}\left(\cos\left(\langle\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\rangle\right),\cos\left(\langle\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\rangle\right)\right).\qquad\text{(A.237)}
\end{aligned}$$

Comparing (A.237) with the equation we are willing to show, there are three main points leaping to the eye. Firstly, the arguments of the companion process differ. Secondly, there is a sum in place of the integral, and lastly, the kernel functions in equation (A.237) have not both the same argument. We address these points in this order as the change of the companion process's argument deletes the dependence from the sum. We have

$$
\begin{aligned}
&\operatorname{Cov}\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\rangle\right), \cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_2\left(\frac{t}{T}\right)\rangle\right)\right)\\
&= \operatorname{Cov}\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h(u)\rangle\right)\right)\\
&\qquad + \operatorname{Cov}\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\rangle\right) - \cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\rangle\right)\right)\\
&\qquad\qquad + \operatorname{Cov}\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\rangle\right) - \cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h(u)\rangle\right)\right)\\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
\tag{A.238}
$$

Because term I is the one we aiming to obtain, we show the negligibility of the last two. Beginning with term II, we can bound the covariance by

$$
\begin{aligned}
&\left|E\left(\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\rangle\right) - \cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right)\right)\cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\rangle\right)\right)\right|\\
&\qquad + \left|E\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\rangle\right) - \cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right)\right) E\cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\rangle\right)\right|.
\end{aligned}
\tag{A.239}
$$

Since the cosine can be bounded by 1, equation (A.239), in turn, can be bounded by

$$
2\,E\left|\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\rangle\right) - \cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right)\right|.
$$

Making use of the Lipschitz condition (5.3) and part (ii) of Lemma 2.3, we obtain

$$
\begin{aligned}
&\left\|\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\rangle\right) - \cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right)\right\|_1\\
&\qquad \le |\underline{s}_1|_1 \left\|\underline{\widetilde{X}}_0\left(\frac{t}{T}\right) - \underline{\widetilde{X}}_0(u)\right\|_1 \le C\left|\frac{t}{T} - u\right| \le C\,b_T
\end{aligned}
$$

where the last step is possible due to the kernel function in (A.237). As b_T tends to 0 as $T \to \infty$, this leads to $\mathrm{II} = o(1)$. Repeating the very same steps for term III of equation (A.238) gives the same result. Furthermore, it holds

$$
\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) K\left(\frac{(t+h)/T - u}{b_T}\right) = \mathcal{O}(1)
$$

following part (ii) of Remark 4.1. In consequence, we can work with

$$
\begin{aligned}
&\lim_{T\to\infty} \operatorname{Cov}\left(\Re\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}_1\right)\right), \Re\left((b_T T)^{1/2}\,\widehat{\varphi}_X\left(u;\underline{s}_2\right)\right)\right)\\
&= \sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) K\left(\frac{(t+h)/T - u}{b_T}\right)\\
&\qquad\qquad \cdot \operatorname{Cov}\left(\cos\left(\langle \underline{s}_1, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}_2, \underline{\widetilde{X}}_h(u)\rangle\right)\right)
\end{aligned}
\tag{A.240}
$$

instead of (A.237). Since the first point, the change of the process' arguments, is finished, we move on to the integral. As we aim to have

$$b_T^{-1}\int_0^1 K\left(\frac{y-u}{b_T}\right)K\left(\frac{y+h/T-u}{b_T}\right)dy \tag{A.241}$$

in place of the sum, we show that the difference between both vanishes with $T\to\infty$. Thus, consider

$$\left|\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)K\left(\frac{(t+h)/T-u}{b_T}\right)-b_T^{-1}\int_0^1 K\left(\frac{y-u}{b_T}\right)K\left(\frac{y+h/T-u}{b_T}\right)dy\right|. \tag{A.242}$$

The equation above can be bounded by

$$b_T^{-1}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{t/T}\left|K\left(\frac{t/T-u}{b_T}\right)K\left(\frac{(t+h)/T-u}{b_T}\right)-K\left(\frac{y-u}{b_T}\right)K\left(\frac{y+h/T-u}{b_T}\right)\right|dy. \tag{A.243}$$

Next, we have a look at the kernel difference. We can rewrite said difference and obtain benefiting from the Lipschitz continuity of the kernel function

$$\begin{aligned}
&\left|K\left(\frac{t/T-u}{b_T}\right)K\left(\frac{(t+h)/T-u}{b_T}\right)-K\left(\frac{y-u}{b_T}\right)K\left(\frac{y+h/T-u}{b_T}\right)\right|\\
&=\left|K\left(\frac{t/T-u}{b_T}\right)\left(K\left(\frac{(t+h)/T-u}{b_T}\right)-K\left(\frac{y+h/T-u}{b_T}\right)\right)\right.\\
&\qquad\qquad\left.+K\left(\frac{y+h/T-u}{b_T}\right)\left(K\left(\frac{t/T-u}{b_T}\right)-K\left(\frac{y-u}{b_T}\right)\right)\right|\\
&\le C\left|\frac{t/T-y}{b_T}\right|\\
&\le \frac{C}{b_T T}.
\end{aligned}$$

Comparable to the calculation earlier, we can perform the last step due to the integral's bounds in equation (A.243). Because of the compact support of the kernel function, this yields $\mathcal{O}\left((b_T T)^{-1}\right)$ as upper bound for the difference in (A.242). In consequence, said difference is negligible as T tends to ∞, and we can focus on (A.241) from now on. Hence, the kernel's arguments are left. Setting $z:=\frac{y-u}{b_T}$, classic substitution leads to

$$b_T^{-1}\int_0^1 K\left(\frac{y-u}{b_T}\right)K\left(\frac{y+h/T-u}{b_T}\right)dy=\int_{-1}^{1}K(z)K\left(z+\frac{h}{b_T T}\right)dy \tag{A.244}$$

for T sufficiently large. Inserting a self-canceling difference turns the right-hand side of (A.244) into

$$\int_{-1}^{1}K^2(z)dy+\int_{-1}^{1}K(z)\left(K\left(z+\frac{h}{b_T T}\right)-K(z)\right)dy. \tag{A.245}$$

Because of the kernel's properties, the integrand belonging to the second integral in (A.245) can be bounded by $C\left|\frac{h}{b_T T}\right|$, which tends to 0 as $T \to \infty$ for fixed h. As a result, it holds

$$\lim_{T\to\infty} b_T^{-1} \int_0^1 K\left(\frac{y-u}{b_T}\right) K\left(\frac{y+h/T-u}{b_T}\right) dy = \int_{-1}^1 K^2(z)dy.$$

Thereupon, the proof is finished. □

Endued with the covariance result, we are now up to demonstrate the CLT designed for the ECF-case. Consequently, this happens in the following proof:

Proof of Theorem 5.4. As in the previous proof, we limit ourselves to the proof of the convergences with regard to the real part. The omitted one can be deduced straightaway. We start by inserting $E\,\widehat{\varphi}_X\left(u;\underline{s}\right)$ with the help of a self-canceling difference, to wit

$$\begin{aligned}
&(b_T T)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)\\
&=(b_T T)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-E\,\widehat{\varphi}_X\left(u;\underline{s}\right)\right)+(b_T T)^{1/2}\,\Re\left(E\,\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)\\
&=:\mathrm{I}+\mathrm{II}. \qquad\qquad (\mathrm{A}.246)
\end{aligned}$$

Now we examine the two terms separately. Following Lemma 4.3, it holds for the first $\mathrm{I}=o(1)$. As opposed to this, we apply version (b) of Theorem 2.15 to the second term of (A.246) and obtain $\mathrm{II} \xrightarrow{d} \mathcal{N}\left(0, V_{X,\Re}\left(u;\underline{s}\right)\right)$ as $T \to \infty$, where the variance is determined by Lemma 5.2. To combine both results, we make use of Slutsky's theorem and get

$$(b_T T)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right) \xrightarrow{d} \mathcal{N}\left(0, V_{X,\Re}\left(u;\underline{s}\right)\right).$$

The application of the Cramér-Wold theorem yields now the desired result using the additivity of the Gaussian distribution. For a detailed consideration of the covariance matrix, we refer to the proof of Theorem 3.1 in Jentsch et al. (2020). □

On our way to show the modified FCLT, the proof of the adjusted tightness result in Lemma 5.5 forms the last stopover. Said proof reads as follows:

Proof of Lemma 5.5. Similar to the proofs before, we focus on the real part case, since the imaginary one can be derived in the very same way.

Once again, we begin by adding self-canceling differences in order to bring the companion process in. This leads to

$$\begin{aligned}
&P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|(b_T T)^{1/2}\,\Re\left(\left(\widehat{\varphi}_X\left(u;\underline{s}_1\right)-\varphi_X\left(u;\underline{s}_1\right)\right)-\left(\widehat{\varphi}_X\left(u;\underline{s}_2\right)-\varphi_X\left(u;\underline{s}_2\right)\right)\right)\right|>\lambda\right)\\
&\leq 2\,P\left(\sup_{\underline{s}\in[-S,S]^d}\left|(b_T T)^{1/2}\,\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right)\right|>\frac{\lambda}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\,P\left(\sup_{\underline{s}\in[-S,S]^d}\left|(b_T T)^{1/2}\,\Re\left(\widehat{\varphi}_X(u;\underline{s}) - E\,\widehat{\varphi}_X(u;\underline{s}) + E\,\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right)\right| > \frac{\lambda}{2}\right)\\
&\leq 2\left(P\left(\sup_{\underline{s}\in[-S,S]^d}\left|(b_T T)^{1/2}\,\Re\left(\widehat{\varphi}_X(u;\underline{s}) - E\,\widehat{\varphi}_X(u;\underline{s})\right)\right| > \frac{\lambda}{4}\right)\right.\\
&\qquad\qquad\left. + P\left(\sup_{\underline{s}\in[-S,S]^d}\left|(b_T T)^{1/2}\,\Re\left(E\,\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right)\right| > \frac{\lambda}{4}\right)\right).
\end{aligned}
$$

Now we aim to show that the second summand can be neglected if T tends to infinity. Using Markov's inequality and Lemma 2.8, we can bound the said summand via

$$
\begin{aligned}
&P\left(\sup_{\underline{s}\in[-S,S]^d}\left|(b_T T)^{1/2}\,\Re\left(E\,\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right)\right| > \frac{\lambda}{4}\right)\\
&\leq \frac{2}{\lambda}\,E\left(\sup_{\underline{s}\in[-S,S]^d}\left|(b_T T)^{1/2}\,\Re\left(E\,\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right)\right|\right)\\
&= o(1)
\end{aligned}
$$

according to Lemma 4.3. This means we are in the same situation as already treated in version (b) of Lemma 2.17. Following the application of said result, we continue with version (b) of Lemma 2.18. This is possible because the finite absolute moments of the function g having the same first and second argument belonging to either the companion process or the truncated companion process can be bounded uniformly. As the $|\cdot|_1$-norm undertakes the role of the function g, the above-described fact is clearly satisfied for the companion process due to stationarity. Regarding the truncated version, Remark 2.2 guarantees the fulfillment. Subsequently, the proof has come to an end. □

Finally, we arrive at the proof of Theorem 5.6, whose shortened form can be found below.

Proof of Theorem 5.6. The proof is made analogously to the one appurtenant to Theorem 2.19 but using Theorem 5.4 as well as Lemma 5.5 in this case. Finally, the use of the Cramér-Wold theorem completes the proof. □

Next, we address a proof of a result newly designed to suit the special case:

Proof of Lemma 5.7. The aim of the steps made in this proof is to be able to use Slutsky's theorem as final step. Therefore, we start by subtracting and adding $\varphi_X(u;\underline{s})$ to its sample counterpart. Then, we split the emerging difference in real and imaginary part. To this end, we get

$$
\begin{aligned}
\widehat{\varphi}_X(u;\underline{s}) &= \widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s}) + \varphi_X(u;\underline{s})\\
&= \frac{1}{(b_T T)^{1/2}}\,(b_T T)^{1/2}\left(\Re\left(\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right) + i\Im\left(\widehat{\varphi}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right)\right)\\
&\qquad + \varphi_X(u;\underline{s})\,.
\end{aligned}
\tag{A.247}
$$

Next, we want to make use of Theorem 5.4. Note that the insertion of $(b_T T)^{-1/2} (b_T T)^{1/2}$ helps us to receive the required prefactor to do so. Since real and imaginary part of equation (A.247) can be treated analogously, we restrict ourselves to the former. Using said theorem, we obtain

$$(b_T T)^{1/2} \Re \left(\widehat{\varphi}_X (u; \underline{s}) - \varphi_X (u; \underline{s})\right) \xrightarrow{d} \mathcal{N} \left(0, V_{X,\Re} (u; \underline{s})\right)$$

with variance $V_{X,\Re} (u; \underline{s})$ having its roots in part (ii) of Remark 5.3. Because of $(b_T T)^{-1/2}$ tending to 0 with $T \to \infty$, we get using Slutsky's theorem

$$\frac{1}{(b_T T)^{1/2}} (b_T T)^{1/2} \Re \left(\widehat{\varphi}_X (u; \underline{s}) - \varphi_X (u; \underline{s})\right) \xrightarrow{d} 0$$

as T tends to ∞. Performing the very same steps, it holds

$$\frac{1}{(b_T T)^{1/2}} (b_T T)^{1/2} \Im \left(\widehat{\varphi}_X (u; \underline{s}) - \varphi_X (u; \underline{s})\right) \xrightarrow{d} 0$$

for $T \to \infty$ as well. Thus, we obtain the desired result using the continuous mapping theorem, namely

$$\widehat{\varphi}_X (u; \underline{s}) \xrightarrow{d} \varphi_X (u; \underline{s})$$

as T tends to ∞. □

Now we return to the general theory and proof the first modified result of Section 5.2, namely Lemma 5.8.

Proof of Lemma 5.8. The result above can be shown by following the lines of the proof belonging to Lemma 2.9 while using the Lipschitz condition (5.3) instead of the former (2.11). □

Subsequently, we use the proof above to demonstrate how to establish a covariance bound regarding Assumption 11 in form of Lemma 5.9.

Proof of Lemma 5.9. W.l.o.g. consider $h \geq 1$. We have a look at the absolute value of the covariance since this is an upper bound. Our first step will be to insert the truncated process $\left(\underline{\widetilde{X}}_t^{(M)}\right)_{t\in\mathbb{Z}}$ as described in equation (2.13) for $M := \lceil h/2 \rceil$ into this bound by adding and subtracting a mixed covariance at the same time. Consequently, we get

$$\begin{aligned}
&\left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0 (u_1)\right), f\left(\underline{s}, \underline{\widetilde{X}}_h (u_2)\right)\right)\right| \\
&\leq \left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0 (u_1)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)} (u_1)\right), f\left(\underline{s}, \underline{\widetilde{X}}_h (u_2)\right)\right)\right| \\
&\qquad + \left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)} (u_1)\right), f\left(\underline{s}, \underline{\widetilde{X}}_h (u_2)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_h^{(M)} (u_2)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II}
\end{aligned}$$

as in the proof of Lemma 2.11. As it can clearly be seen, terms I and II have the same building type. Hence, we focus on the first one. We have a closer look at term I and get

$$\begin{aligned} \mathrm{I} &\leq 2\, C_{f,2}\, E \left| f\left(\underline{s}, \underline{\widetilde{X}}_0\left(u_1\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_0^{(M)}\left(u_1\right)\right)\right| \\ &\leq C\, |\underline{s}|_1\, \|\varepsilon_0\|_1\, d \sum_{|j|\geq M} \frac{B}{l(j)} \end{aligned}$$

using Lemma 5.8. Again, by repeating the same steps for term II we obtain the very same upper bound

$$\left|\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0\left(u_1\right)\right), f\left(\underline{s}, \underline{\widetilde{X}}_h\left(u_2\right)\right)\right)\right| \leq C\, |\underline{s}|_1\, \|\varepsilon_0\|_1\, d \sum_{|j|\geq M} \frac{B}{l(j)} \leq |\underline{s}|_1 \frac{C_{Cov,cs}}{h^{1+\tilde{\delta}}}$$

performing the same steps as in part (b) of the proof apurtenant to Lemma 2.11. This completes the proof. □

The next proof takes us back to the ECF-case at it belongs to Lemma 5.11.

Proof of Lemma 5.11. The first step will be to insert $\widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right)$ defined in (5.5) into the difference in question using a self-canceling difference. Thus, we get

$$E\left|\widehat{\varphi}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right|^2 \leq E\left(\left|\widehat{\varphi}_X\left(u;\underline{s}\right) - \widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right)\right| + \left|\widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right|\right)^2. \tag{A.248}$$

By expanding the quadratic sum in (A.248), we obtain

$$E\left|\widehat{\varphi}_X\left(u;\underline{s}\right) - \widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right)\right|^2 + E\left|\widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right|^2 =: \mathrm{I} + \mathrm{II} \tag{A.249}$$

as an upper bound for equation (A.248) disregarding multiplying constants to work with instead. Starting with term I, it holds

$$\mathrm{I} = E\left(\Re\left(\widehat{\varphi}_X\left(u;\underline{s}\right) - \widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right)\right)\right)^2 + E\left(\Im\left(\widehat{\varphi}_X\left(u;\underline{s}\right) - \widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right)\right)\right)^2 =: \mathrm{Ia} + \mathrm{Ib}. \tag{A.250}$$

Because the difference between subterm Ia and Ib consists of the use of the cosine function in the first case and the sine function in the other, we confine ourselves to the examination of subterm Ia. Afterwards, the results can easily be transferred to the remaining case. We have

$$\begin{aligned} \mathrm{Ia} &= E\left(\Re\,\widehat{\varphi}_X\left(u;\underline{s}\right) - \Re\,\widehat{\widetilde{\varphi}}_X\left(u;\underline{s}\right)\right)^2 \\ &= E\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(\langle \underline{s}, \underline{X}_{t,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t(u)\rangle\right)\right)\right)^2 \end{aligned}$$

$$\leq \frac{1}{(b_T T)^2} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right)$$
$$\cdot E\left|\left(\cos\left(\langle \underline{s}, \underline{X}_{t_1,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1}(u)\rangle\right)\right)\left(\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right)\right|. \tag{A.251}$$

Since we are interested in making use of the closeness between the process and its companion counterpart, we apply Hoelder's inequality in order to obtain a suitable exponent corresponding to the number of finite absolute moments the innovations possess. This leads to

$$\mathrm{Ia} \leq \frac{1}{(b_T T)^2} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right)$$
$$\cdot \left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_1,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1}(u)\rangle\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}$$
$$\cdot \left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right|^{\frac{2+\delta}{\delta}}\right)^{\frac{\delta}{2+\delta}} \tag{A.252}$$

in place of (A.251) to move on with. Before we continue with the examination of subterm Ia, we turn our attention to the differences of the cosine functions. Since both differences only differ in terms of the index, we focus on the first one. To begin, the Lipschitz condition (5.3) gives

$$\left|\cos\left(\langle \underline{s}, \underline{X}_{t_1,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1}(u)\rangle\right)\right| \leq |\underline{s}|_1 \left|\underline{X}_{t_1,T} - \underline{\widetilde{X}}_{t_1}(u)\right|_1 .$$

Now we go back to equation (A.252) and insert our result into the first factor originating from the use of Hoelder's inequality. Thus, we have

$$\left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_1,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1}(u)\rangle\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} \leq C\,|\underline{s}|_1 \left\|\underline{X}_{t_1,T} - \underline{\widetilde{X}}_{t_1}(u)\right\|_{\frac{2+\delta}{2}} .$$

Since we have results for the closeness between $\underline{X}_{t_1,T}$ and $\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)$ as well as $\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)$ and $\underline{\widetilde{X}}_{t_1}(u)$, the next step is to insert the missing version of the companion process, that is

$$|\underline{s}|_1 \left\|\underline{X}_{t_1,T} - \underline{\widetilde{X}}_{t_1}(u)\right\|_{\frac{2+\delta}{2}} \leq |\underline{s}|_1 \left(\left\|\underline{X}_{t_1,T} - \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right\|_{\frac{2+\delta}{2}} + \left\|\underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right) - \underline{\widetilde{X}}_{t_1}(u)\right\|_{\frac{2+\delta}{2}}\right).$$

With the help of Lemma 2.3, we establish the following bound

$$\begin{aligned}\left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_1,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1}(u)\rangle\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} &\leq |\underline{s}|_1 \left(\frac{C_B}{T} + C_{\bar{B}}\left|\frac{t_1}{T} - u\right|\right)\\ &\leq C_1\,|\underline{s}|_1 \left(T^{-1} + b_T\right)\\ &\leq C_1\,|\underline{s}|_1\, b_T.\end{aligned}$$

The last steps follow the same argumentation as in the proof of Lemma 4.3. Now we tend to the remaining factor containing the second difference in (A.252). Because the innovations come only with finite absolute moments of order $\frac{2+\delta}{2}$, we need to divide the exponent. Therefore, we get

$$\begin{aligned}
&\left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right)-\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right|^{\frac{2+\delta}{\delta}}\right)^{\frac{\delta}{2+\delta}}\\
&=\left(E\left(\left|\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right)-\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right|^{\frac{2+\delta}{2}}\right.\right.\\
&\qquad\qquad\left.\left.\cdot\left|\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right)-\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right|^{\frac{4-\delta^2}{2\delta}}\right)\right)^{\frac{\delta}{2+\delta}}.
\end{aligned}\tag{A.253}$$

As the cosine function can be bounded by 1, we can work with

$$\left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right)-\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{\delta}{2+\delta}}$$

in lieu of (A.253). Following the same steps as before, we obtain

$$\begin{aligned}
&\left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right)-\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{\delta}{2+\delta}}\\
&=\left(\left(E\left|\cos\left(\langle \underline{s}, \underline{X}_{t_2,T}\rangle\right)-\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}\right)^{\delta/2}\\
&\leq C_2\left(|\underline{s}|_1\, b_T\right)^{\delta/2}.
\end{aligned}$$

Combining both bounds, we get returning to equation (A.252)

$$\begin{aligned}
\mathrm{Ia} &\leq \frac{C_3}{(b_T T)^2}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right)K\left(\frac{t_2/T-u}{b_T}\right)|\underline{s}|_1\, b_T\left(|\underline{s}|_1\, b_T\right)^{\delta/2}\\
&=|\underline{s}|_1^{\frac{2+\delta}{2}}\,\mathcal{O}\left(b_T^{\frac{2+\delta}{2}}\right)
\end{aligned}$$

having Assumption 9 in mind. Repeating the same steps for Ib of equation (A.250) leads to the same result. In conclusion, $\mathrm{I}=|\underline{s}|_1^{\frac{2+\delta}{2}}\,\mathcal{O}\left(b_T^{\frac{2+\delta}{2}}\right)$ holds true as well. Now we turn to the second term of (A.249). First, we decompose the squared absolute value as before to get

$$\mathrm{II}=E\left(\Re\left(\widehat{\varphi}_X(u;\underline{s})-\varphi_X(u;\underline{s})\right)\right)^2+E\left(\Im\left(\widehat{\varphi}_X(u;\underline{s})-\varphi_X(u;\underline{s})\right)\right)^2=:\mathrm{IIa}+\mathrm{IIb}.\tag{A.254}$$

Drawing on the same argumentation as earlier, we proceed with the detailed examination of only the first subterm of (A.254). We rewrite said term in order to create a centered difference, to wit

$$
\begin{aligned}
\mathrm{IIa} &= E\left(\Re\,\widehat{\varphi}_X\left(u;\underline{s}\right) - \Re\,\varphi_X\left(u;\underline{s}\right)\right)^2 \\
&= E\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right) - E\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right)\right)\right. \\
&\qquad \left. - \left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) - 1\right) E\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right)\right)^2 .
\end{aligned}
$$

This expression can easily be bounded by

$$
\begin{aligned}
&E\left(\left|\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right) - E\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right)\right)\right|\right. \\
&\qquad \left. + \left|\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) - 1\right) E\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right)\right|\right)^2 . \qquad \text{(A.255)}
\end{aligned}
$$

Taking the square of each summand of equation (A.255) bequeaths us the bound below while ignoring the associated constants:

$$
\begin{aligned}
&E\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right) - E\cos\left(\langle\underline{s},\underline{\widetilde{X}}_t(u)\rangle\right)\right)\right)^2 \\
&\qquad + \left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) - 1\right)^2 \\
&=: \mathrm{IIaa} + \mathrm{IIab}. \qquad \text{(A.256)}
\end{aligned}
$$

In the following, the two newly defined subterms are treated separately. We will start with the first, namely IIaa. Afterwards, we will have a look at the last subterm. Using the stationarity of the companion process, we can rewrite IIaa in the following way

$$
\begin{aligned}
\mathrm{IIaa} &= \frac{C}{(b_T T)^2}\sum_{t_1=1}^{2\lfloor b_T T\rfloor+1}\sum_{t_2=1}^{2\lfloor b_T T\rfloor+1} E\left(\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_1}(u)\rangle\right) - E\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_1}(u)\rangle\right)\right)\right. \\
&\qquad \left. \cdot\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_2}(u)\rangle\right) - E\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right)\right) \\
&= \frac{C}{(b_T T)^2}\sum_{t_1=1}^{2\lfloor b_T T\rfloor+1}\sum_{t_2=1}^{2\lfloor b_T T\rfloor+1} \mathrm{Cov}\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_1}(u)\rangle\right),\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_2}(u)\rangle\right)\right) \\
&= \frac{C}{(b_T T)^2}\sum_{t_1=1}^{2\lfloor b_T T\rfloor+1}\sum_{t_2=1}^{2\lfloor b_T T\rfloor+1} \mathrm{Cov}\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_1-t_2}(u)\rangle\right),\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{0}(u)\rangle\right)\right). \qquad \text{(A.257)}
\end{aligned}
$$

Now, with the help of Lemma 5.9 and the fact that it holds

$$\left|\operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right)\right)\right| \leq 4, \tag{A.258}$$

we can bound equation (A.257) via

$$\begin{aligned}
&\frac{C}{(b_T T)^2} \sum_{t=-2\lfloor b_T T\rfloor}^{2\lfloor b_T T\rfloor} (2\lfloor b_T T\rfloor + 1 - |t|) \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right)\right) \\
&= \frac{C}{(b_T T)^2} \Bigg(2 \sum_{t=1}^{2\lfloor b_T T\rfloor} (2\lfloor b_T T\rfloor + 1 - t) \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_t(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right)\right) \\
&\qquad\qquad + (2\lfloor b_T T\rfloor + 1) \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right)\right)\Bigg) \\
&\leq \frac{C}{(b_T T)^2} \left(2 \sum_{t=1}^{2\lfloor b_T T\rfloor} (2\lfloor b_T T\rfloor + 1 - t) |\underline{s}|_1 \frac{C_{Cov,cs}}{t^{1+\tilde{\delta}}} + 4(2\lfloor b_T T\rfloor + 1)\right) \\
&= (|\underline{s}|_1 + 1)\, \mathcal{O}\left(\frac{1}{b_T T}\right).
\end{aligned}$$

Next, we move on to subterm IIab of equation (A.256). According to Lemma 4.2, it holds

$$\left|\frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) - 1\right| = \mathcal{O}\left(\frac{1}{b_T T}\right),$$

and since in subterm IIab the difference stated above is squared, we have

$$\mathrm{IIab} = \mathcal{O}\left(\frac{1}{(b_T T)^2}\right).$$

Combining both subterms and keeping in mind that the very same bounds apply for subterm IIb as well, we obtain

$$\mathrm{II} = (|\underline{s}|_1 + 1)\, \mathcal{O}\left(\frac{1}{b_T T}\right) + \mathcal{O}\left(\frac{1}{(b_T T)^2}\right) = (|\underline{s}|_1 + 1)\, \mathcal{O}\left(\frac{1}{b_T T}\right).$$

Recapitulatory, it holds

$$E\left|\widehat{\varphi}_X(u; \underline{s}) - \varphi_X(u; \underline{s})\right|^2 = |\underline{s}|_1^{\frac{2+\delta}{2}}\, \mathcal{O}\left(b_T^{\frac{2+\delta}{2}}\right) + (|\underline{s}|_1 + 1)\, \mathcal{O}\left((b_T T)^{-1}\right),$$

which is the result we were aiming to show. □

With the last proof, the preparatory part is closed, and we are able to approach the consistency proof in the following lines. The proof is inspired by the proof belonging to Theorem 4.2 in Jentsch et al. (2020).

Proof of Theorem 5.13. First, note that

$$\widehat{\mathfrak{C}}_{Y,Z}(u) \xrightarrow{P} 0$$

as $T \to \infty$ is equivalent to

$$\widehat{\mathfrak{C}}_{Y,Z}(u) \xrightarrow{d} 0$$

as $T \to \infty$ because 0 is a constant. This allows us to use the same technique as seen in the proof of Theorem 2.15. Thus, we need to show the fulfillment of the following conditions:

(1) For $\eta \in (0,1)$ and

$$D_\eta := \left\{ (\underline{s}_1', \underline{s}_1')' \,\middle|\, \eta \leq |\underline{s}_1|_2 \,, |\underline{s}_2|_2 \leq 1/\eta \right\},$$

it holds

$$\widehat{\mathfrak{C}}_{Y,Z;\eta}(u) := \int_{D_\eta} \left| \widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right) \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \right|^2 d\mathfrak{w} \xrightarrow{d} 0$$

as T tends to ∞.

(2) For $\eta \to 0$, we have

$$\mathfrak{C}_{Y,Z;\eta}(u) \longrightarrow 0.$$

(3) For all $\epsilon > 0$, it holds

$$\lim_{\eta \downarrow 0} \limsup_{T \to \infty} P\left(\left| \widehat{\mathfrak{C}}_{Y,Z;\eta}(u) - \widehat{\mathfrak{C}}_{Y,Z}(u) \right| > \epsilon \right) = 0.$$

Then, similar to the proof of Theorem 2.15 the sought-after convergence can be deduced straight away using Proposition 6.3.9 of Brockwell and Davis (1991). We start with requirement (1). As first step, we rewrite the difference due to the independence of $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$ in the following way:

$$\begin{aligned}
&\widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right) \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \\
&= \widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \varphi_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) + \varphi_Y\left(u; \underline{s}_1\right) \varphi_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right) \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \\
&= \widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \varphi_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) + \varphi_Y\left(u; \underline{s}_1\right) \left(\varphi_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}_Z\left(u; \underline{s}_2\right)\right) \\
&\qquad + \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \left(\varphi_Y\left(u; \underline{s}_1\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right)\right). \qquad \text{(A.259)}
\end{aligned}$$

This leads to

$$\begin{aligned}
\widehat{\mathfrak{C}}_{Y,Z;\eta}(u) = \int_{D_\eta} &\left| \widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \varphi_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) + \varphi_Y\left(u; \underline{s}_1\right) \left(\varphi_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}_Z\left(u; \underline{s}_2\right)\right) \right. \\
&\left. + \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \left(\varphi_Y\left(u; \underline{s}_1\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right)\right) \right|^2 d\mathfrak{w}. \qquad \text{(A.260)}
\end{aligned}$$

Now Lemma 5.7 in combination with Slutsky's theorem tells us that the second and the third summand of the integrand in (A.260) tend in distribution to 0 as T tends to ∞. In this context, note that both $\varphi_{Y,Z}(u;\underline{s}_1,\underline{s}_2)$ and $\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2)$ are built the same way as, for example, $\varphi_Y(u;\underline{s}_1)$ and its sample counterpart $\widehat{\varphi}_Y(u;\underline{s}_1)$ since the product of exponential functions can be merged into one by summing up the exponents. Hence, using

$$\underline{X}_{t,T} = \left(\underline{Y}_{t,T}', \underline{Z}_{t,T}'\right)'$$

and defining

$$\underline{s} := \left(\underline{s}_1', \underline{s}_2'\right)' \tag{A.261}$$

combined with the use of the continuous mapping theorem leads to

$$\widehat{\mathfrak{C}}_{Y,Z;\eta}(u) \xrightarrow{d} 0$$

as T tends to ∞. This finishes (1), and we move on to (2). As a first step, we rewrite $\widehat{\mathfrak{C}}_{Y,Z;\eta}(u)$ the following way:

$$\widehat{\mathfrak{C}}_{Y,Z;\eta}(u) = \int_{\mathbb{R}^p\times\mathbb{R}^q} \mathbb{1}_{D_\eta}\left(\left(\underline{s}_1', \underline{s}_1'\right)'\right) \left|\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_Y(u;\underline{s}_1)\,\widehat{\varphi}_Z(u;\underline{s}_2)\right|^2 d\mathfrak{w}.$$

Now the integrand above forms a sequence in η converging to

$$\left|\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_Y(u;\underline{s}_1)\,\widehat{\varphi}_Z(u;\underline{s}_2)\right|^2$$

as η tends to 0. Besides, the limiting difference can be bounded by 4. This allows for the use of the dominated convergence theorem. Thereby, we obtain

$$\begin{aligned}\lim_{\eta\downarrow 0}\widehat{\mathfrak{C}}_{Y,Z;\eta}(u) &= \int_{\mathbb{R}^p\times\mathbb{R}^q} \lim_{\eta\downarrow 0}\mathbb{1}_{D_\eta}\left(\left(\underline{s}_1', \underline{s}_1'\right)'\right) \left|\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_Y(u;\underline{s}_1)\,\widehat{\varphi}_Z(u;\underline{s}_2)\right|^2 d\mathfrak{w}\\ &= \int_{\mathbb{R}^p\times\mathbb{R}^q} \left|\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_Y(u;\underline{s}_1)\,\widehat{\varphi}_Z(u;\underline{s}_2)\right|^2 d\mathfrak{w}\\ &= \widehat{\mathfrak{C}}_{Y,Z}(u),\end{aligned}$$

which terminates the verification of the second requirement. Next, we move on to constraint (3). Applying Markov's inequality and Fubini's theorem leads to

$$\begin{aligned}&P\left(\left|\widehat{\mathfrak{C}}_{Y,Z;\eta}(u) - \widehat{\mathfrak{C}}_{Y,Z}(u)\right| > \epsilon\right)\\ &= P\left(\left|\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} \left|\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_Y(u;\underline{s}_1)\,\widehat{\varphi}_Z(u;\underline{s}_2)\right|^2 d\mathfrak{w}\right| > \epsilon\right)\\ &\le \epsilon^{-1}E\left(\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} \left|\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_Y(u;\underline{s}_1)\,\widehat{\varphi}_Z(u;\underline{s}_2)\right|^2 d\mathfrak{w}\right)\\ &= \epsilon^{-1}\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} E\left(\left|\widehat{\varphi}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_Y(u;\underline{s}_1)\,\widehat{\varphi}_Z(u;\underline{s}_2)\right|^2\right) d\mathfrak{w}.\end{aligned} \tag{A.262}$$

Now we expand and bound the integrand in (A.262) as follows:

$$\begin{aligned}
&E\left(\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2\right)\\
&\leq E\left(\left(\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|+\left|\varphi_Y\left(u;\underline{s}_1\right)\left(\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\right|\right.\right.\\
&\qquad\left.\left.+\left|\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\left(\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right)\right|\right)^2\right),\\
&\leq E\left(\left(\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|+\left|\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|\right.\right.\\
&\qquad\left.\left.+\left|\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right|\right)^2\right)
\end{aligned} \tag{A.263}$$

because the absolute value of the ECF is bounded by 1. Regardless of any constants, an upper bound for (A.263) is provided by

$$\begin{aligned}
&E\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2+E\left|\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2\\
&\qquad+E\left|\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right|^2\\
&=: \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned} \tag{A.264}$$

We see that the three terms of (A.264) are built in the same manner. Thus, the way we establish upper bounds for these terms are similar. In order to establish these upper bounds, we want to make use of Lemma 5.11. Starting with term I, we get using

$$|\underline{s}|_1=|\underline{s}_1|_1+|\underline{s}_2|_1$$

for $\underline{s}$ defined as in (A.261) the first upper bound, namely

$$\begin{aligned}
\mathrm{I}&=|\underline{s}|_1^{\frac{2+\delta}{2}}\,\mathcal{O}\left(b_T^{\frac{2+\delta}{2}}\right)+\left(|\underline{s}|_1+1\right)\mathcal{O}\left(\left(b_T T\right)^{-1}\right)\\
&=\left(|\underline{s}|_1^{\frac{2+\delta}{2}}+1\right)o(1)\\
&=\left(|\underline{s}_1|_1^{\frac{2+\delta}{2}}+|\underline{s}_2|_1^{\frac{2+\delta}{2}}+1\right)o(1).
\end{aligned}$$

Analogously, it holds

$$\mathrm{II}=\left(|\underline{s}_2|_1^{\frac{2+\delta}{2}}+1\right)o(1)\qquad\text{and}\qquad\mathrm{III}=\left(|\underline{s}_1|_1^{\frac{2+\delta}{2}}+1\right)o(1).$$

The combination of those three upper bounds allows us to bound (A.263) via

$$E\left(\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2\right)=o(1)\left(|\underline{s}_1|_1^{\frac{2+\delta}{2}}+|\underline{s}_2|_1^{\frac{2+\delta}{2}}+1\right).$$

Note that in the calculations above determining the upper bounds, we did not make further use of the independence of $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$ because is would not have any effect on the established bounds. Returning to equation (A.262) as a whole, we replace the integrand by the newly determined upper bound and get

$$P\left(\left|\widehat{\mathfrak{C}}_{Y,Z;\eta}(u)-\widehat{\mathfrak{C}}_{Y,Z}(u)\right|>\epsilon\right)=o(1)\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta}\left(|\underline{s}_1|_1^{\frac{2+\delta}{2}}+|\underline{s}_2|_1^{\frac{2+\delta}{2}}+1\right)d\mathfrak{w},$$

which tends to 0 as $T\to\infty$ because of Assumption 10. This finishes the proof. □

At this point, the first subsection of proofs belonging to Chapter 5 is brought to an end.

A.4.2. Proofs of Section 5.3

This subsection addresses the remaining section of Chapter 5, that is Section 5.3.

After having shown the validity of Theorem 5.6 in the previous section, we make use of it now to prove another, namely Theorem 5.14. Again, we took inspiration of a proof which can be found in Jentsch et al. (2020), namely the one belonging to Theorem 4.3. This proof, in turn, was based on the proof of Theorem 3.2(2) in Davis et al. (2018).

Proof of Theorem 5.14. Similar to the proof of Theorem 5.13, we need to verify certain conditions in order to prove

$$\begin{aligned} b_T T \, \widehat{\mathfrak{C}}_{Y,Z}(u) &= b_T T \int_{\mathbb{R}^p \times \mathbb{R}^q} \left| \widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right) \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \right|^2 d\mathfrak{w} \\ &\xrightarrow{d} \int_{\mathbb{R}^p \times \mathbb{R}^q} \left| G\left(u; \underline{s}_1, \underline{s}_2\right) \right|^2 d\mathfrak{w}. \end{aligned}$$

The conditions mentioned above are in this case the following:

(1) For $\eta \in (0,1)$ and

$$D_\eta := \left\{ (\underline{s}_1', \underline{s}_1')' \,\middle|\, \eta \leq \left|\underline{s}_1\right|_2, \left|\underline{s}_2\right|_2 \leq 1/\eta \right\},$$

it holds

$$\begin{aligned} b_T T \, \widehat{\mathfrak{C}}_{Y,Z;\eta}(u) &:= b_T T \int_{D_\eta} \left| \widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right) \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \right|^2 d\mathfrak{w} \\ &\xrightarrow{d} \int_{D_\eta} \left| G\left(u; \underline{s}_1, \underline{s}_2\right) \right|^2 d\mathfrak{w} \end{aligned}$$

as T tends to ∞.

(2) For $\eta \to 0$, we have

$$\mathcal{G}_\eta := \int_{D_\eta} \left| G\left(u; \underline{s}_1, \underline{s}_2\right) \right|^2 d\mathfrak{w} \xrightarrow{d} \int_{\mathbb{R}^p \times \mathbb{R}^q} \left| G\left(u; \underline{s}_1, \underline{s}_2\right) \right|^2 d\mathfrak{w} =: \mathcal{G}.$$

(3) For all $\epsilon > 0$, it holds

$$\lim_{\eta \downarrow 0} \limsup_{T \to \infty} P\left(\left| b_T T \int_{\mathbb{R}^p \times \mathbb{R}^q \setminus D_\eta} \left| \widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right) \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \right|^2 d\mathfrak{w} \right| > \epsilon \right) = 0.$$

Then, as seen in the proof of Theorem 5.13 Proposition 6.3.9 in Brockwell and Davis (1991) allows us to deduce the sought-after convergence straight away. We begin with (1) and, analogously to (A.259) in said proof, we rewrite the difference in question while making use of the independence of $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$ as follows:

$$\begin{aligned}
&\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\\
&=\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)+\varphi_Y\left(u;\underline{s}_1\right)\left(\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\\
&\qquad+\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\left(\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right). \qquad \text{(A.265)}
\end{aligned}$$

Using this expansion, we can rewrite $\widehat{\mathfrak{C}}_{Y,Z;\eta}(u)$ as well. Hence, we obtain

$$\begin{aligned}
&b_T T\,\widehat{\mathfrak{C}}_{Y,Z;\eta}(u)\\
&=\int_{D_\eta}\Big|\left(b_T T\right)^{1/2}\left(\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\\
&\qquad+\left(b_T T\right)^{1/2}\varphi_Y\left(u;\underline{s}_1\right)\left(\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\\
&\qquad+\left(b_T T\right)^{1/2}\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\left(\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right)\Big|^2 d\mathfrak{w}.
\end{aligned}$$

The next step will be to divide the integrand from above to some extend into real and imaginary parts. In doing so, we get

$$\begin{aligned}
&b_T T\,\widehat{\mathfrak{C}}_{Y,Z;\eta}(u)\\
&=\int_{D_\eta}\Big|\left(b_T T\right)^{1/2}\Re\left(\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\\
&\qquad+i\left(b_T T\right)^{1/2}\Im\left(\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\\
&\qquad+\left(b_T T\right)^{1/2}\varphi_Y\left(u;\underline{s}_1\right)\Re\left(\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\\
&\qquad+i\left(b_T T\right)^{1/2}\varphi_Y\left(u;\underline{s}_1\right)\Im\left(\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\\
&\qquad+\left(b_T T\right)^{1/2}\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\Re\left(\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right)\\
&\qquad+i\left(b_T T\right)^{1/2}\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\Im\left(\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right)\Big|^2 d\mathfrak{w}. \qquad \text{(A.266)}
\end{aligned}$$

Now we have a look at the individual summands of the integrand above. As D_η is a compact space, we can make use of Theorem 5.6. Having the definition of the Gaussian processes directly stated in the theorem in mind, it holds

$$\left(b_T T\right)^{1/2}\varphi_Y\left(u;\underline{s}_1\right)\Re\left(\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\xrightarrow{d}\varphi_Y\left(u;\underline{s}_1\right)G_{Z,\Re}\left(u;\underline{s}_2\right)$$

and

$$\left(b_T T\right)^{1/2}\varphi_Y\left(u;\underline{s}_1\right)\Im\left(\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\xrightarrow{d}\varphi_Y\left(u;\underline{s}_1\right)G_{Z,\Im}\left(u;\underline{s}_2\right)$$

for $T\to\infty$. In combination with Lemma 5.7 and Slutsky's theorem, the application of Theorem 5.6 leads to

$$(b_T T)^{1/2}\, \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \Re\left(\varphi_Y\left(u; \underline{s}_1\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right)\right) \xrightarrow{d} \varphi_Z\left(u; \underline{s}_2\right) G_{Y,\Re}\left(u; \underline{s}_1\right)$$

and

$$(b_T T)^{1/2}\, \widehat{\varphi}_Z\left(u; \underline{s}_2\right) \Im\left(\varphi_Y\left(u; \underline{s}_1\right) - \widehat{\varphi}_Y\left(u; \underline{s}_1\right)\right) \xrightarrow{d} \varphi_Z\left(u; \underline{s}_2\right) G_{Y,\Im}\left(u; \underline{s}_1\right)$$

as T tends to ∞ as well. Lastly, we remember from the proof of Theorem 5.13 that both $\varphi_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right)$ and $\widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right)$ can be treated in a similar way to, for example, $\varphi_Y\left(u; \underline{s}_1\right)$ and $\varphi_Y\left(u; \underline{s}_1\right)$, respectively, using

$$\underline{X}_{t,T} = \left(\underline{Y}'_{t,T}, \underline{Z}'_{t,T}\right)' \quad \text{and} \quad \underline{s} := \left(\underline{s}'_1, \underline{s}'_2\right)'. \tag{A.267}$$

That being said, we rewrite the remaining differences in equation (A.266) and obtain

$$(b_T T)^{1/2}\, \Re\left(\widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \varphi_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right)\right) = (b_T T)^{1/2}\, \Re\left(\widehat{\varphi}_X\left(u; \underline{s}\right) - \varphi_X\left(u; \underline{s}\right)\right)$$

and

$$(b_T T)^{1/2}\, \Im\left(\widehat{\varphi}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - \varphi_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right)\right) = (b_T T)^{1/2}\, \Im\left(\widehat{\varphi}_X\left(u; \underline{s}\right) - \varphi_X\left(u; \underline{s}\right)\right),$$

respectively. Hence, once again, Theorem 5.6 combined with Slutsky's theorem gives

$$(b_T T)^{1/2}\, \Re\left(\widehat{\varphi}_X\left(u; \underline{s}\right) - \varphi_X\left(u; \underline{s}\right)\right) \xrightarrow{d} G_{X,\Re}\left(u; \underline{s}\right)$$

and

$$(b_T T)^{1/2}\, \Im\left(\widehat{\varphi}_X\left(u; \underline{s}\right) - \varphi_X\left(u; \underline{s}\right)\right) \xrightarrow{d} G_{X,\Im}\left(u; \underline{s}\right)$$

as $T \to \infty$. Note that the limiting processes are all Gaussian processes whose sum forms again a Gaussian process. Returning to (A.266), the results we obtained above combined with the continuous mapping theorem lead to the sought-after convergence in distribution, that is

$$b_T T\, \widehat{\mathfrak{C}}_{Y,Z;\eta}(u) \xrightarrow{d} \int_{D_\eta} \left|G\left(u; \underline{s}_1, \underline{s}_2\right)\right|^2 d\mathfrak{w}$$

as T tends to ∞. Since $\left(G\left(u; \underline{s}_1, \underline{s}_2\right)\right)_{(\underline{s}_1, \underline{s}_2) \in \mathbb{R}^p \times \mathbb{R}^q}$ is a centered Gaussian process which is composed of the Gaussian processes having their roots in the convergences of the summands of the integrand of (A.266), it holds

$$\begin{aligned}
&G\left(u; \underline{s}_2 \underline{s}_1, \right) \\
&= G_{X,\Re}\left(u; \underline{s}\right) + i G_{X,\Im}\left(u; \underline{s}\right) + \Re\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Re}\left(u; \underline{s}_2\right) + i\,\Im\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Re}\left(u; \underline{s}_2\right) \\
&\qquad + i\,\Re\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Im}\left(u; \underline{s}_2\right) - \Im\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Im}\left(u; \underline{s}_2\right) + \Re\,\varphi_Z\left(u; \underline{s}_2\right) G_{Y,\Re}\left(u; \underline{s}_1\right) \\
&\qquad\quad + i\,\Im\,\varphi_Z\left(u; \underline{s}_2\right) G_{Z,\Re}\left(u; \underline{s}_1\right) + i\,\Re\,\varphi_Z\left(u; \underline{s}_2\right) G_{Y,\Im}\left(u; \underline{s}_1\right) - \Im\,\varphi_Z\left(u; \underline{s}_2\right) G_{Z,\Im}\left(u; \underline{s}_1\right) \\
&= G_{X,\Re}\left(u; \underline{s}\right) + \Re\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Re}\left(u; \underline{s}_2\right) - \Im\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Im}\left(u; \underline{s}_2\right) \\
&\qquad\qquad + \Re\,\varphi_Z\left(u; \underline{s}_2\right) G_{Y,\Re}\left(u; \underline{s}_1\right) - \Im\,\varphi_Z\left(u; \underline{s}_2\right) G_{Z,\Im}\left(u; \underline{s}_1\right) \\
&\qquad + i\left(G_{X,\Im}\left(u; \underline{s}\right) + \Im\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Re}\left(u; \underline{s}_2\right) + \Re\,\varphi_Y\left(u; \underline{s}_1\right) G_{Z,\Im}\left(u; \underline{s}_2\right)\right. \\
&\qquad\qquad\qquad \left. + \Im\,\varphi_Z\left(u; \underline{s}_2\right) G_{Z,\Re}\left(u; \underline{s}_1\right) + \Re\,\varphi_Z\left(u; \underline{s}_2\right) G_{Y,\Im}\left(u; \underline{s}_1\right)\right)
\end{aligned}$$

$$=: G_{\Re}\left(u; \underline{s}_1, \underline{s}_2\right) + i\, G_{\Im}\left(u; \underline{s}_1, \underline{s}_2\right). \tag{A.268}$$

Thus, the corresponding covariance function is based on those of the previously determined limiting Gaussian processes as well. This ends part (1) and we move on to constraint (2). Following the argumentation in Jentsch et al. (2020), we note that (1) and (3) guarantee the existence of the limit distribution of $b_T T\, \widehat{\mathfrak{C}}_{Y,Z}(u)$. This can be seen in Theorem 2 in Dehling et al. (2009). So (2) has the purpose to ensure the desired form of the limit distribution. Since convergence in the $\mathcal{L}_1$ sense implies convergence in distribution, we aim to show

$$E\left|\mathcal{G}_\eta - \mathcal{G}\right| \longrightarrow 0$$

as η tends to 0. In the beginning, we need to verify the existence of the first moments of $\mathcal{G}_\eta$ and $\mathcal{G}$. We start with the former and get with the use of Fubini's theorem the following:

$$E\,\mathcal{G}_\eta = E \int_{D_\eta} \left|G\left(u; \underline{s}_1, \underline{s}_2\right)\right|^2 d\mathfrak{w} = \int_{D_\eta} E\left|G\left(u; \underline{s}_1, \underline{s}_2\right)\right|^2 d\mathfrak{w}. \tag{A.269}$$

Now we turn our attention to the Gaussian process originating from (1) and continue with its further examination in order to bound (A.269). Since $G\left(u; \underline{s}_1, \underline{s}_2\right)$ is centered, it holds for the integrand

$$\begin{aligned} V_G\left(u; \underline{s}_1, \underline{s}_2\right) &:= E\left|G\left(u; \underline{s}_1, \underline{s}_2\right)\right|^2 \\ &= E\left(G_{\Re}\left(u; \underline{s}_1, \underline{s}_2\right)\right)^2 + E\left(G_{\Im}\left(u; \underline{s}_1, \underline{s}_2\right)\right)^2 \\ &= \operatorname{Var}\left(G_{\Re}\left(u; \underline{s}_1, \underline{s}_2\right)\right) + \operatorname{Var}\left(G_{\Im}\left(u; \underline{s}_1, \underline{s}_2\right)\right). \end{aligned} \tag{A.270}$$

Therefore, the next step will be to specify $\operatorname{Var}\left(G_{\Re}\left(u; \underline{s}_1, \underline{s}_2\right)\right)$ and $\operatorname{Var}\left(G_{\Im}\left(u; \underline{s}_1, \underline{s}_2\right)\right)$ starting with the former. We have

$$\begin{aligned} &\operatorname{Var}\left(G_{\Re}\left(u; \underline{s}_1, \underline{s}_2\right)\right) \\ &= V_{X,\Re}\left(u; \underline{s}\right) + \left(\Re\,\varphi_Y\left(u; \underline{s}_1\right)\right)^2 V_{Z,\Re}\left(u; \underline{s}_2\right) + \left(\Im\,\varphi_Y\left(u; \underline{s}_1\right)\right)^2 V_{Z,\Im}\left(u; \underline{s}_2\right) \\ &\qquad + \left(\Re\,\varphi_Z\left(u; \underline{s}_2\right)\right)^2 V_{Y,\Re}\left(u; \underline{s}_1\right) + \left(\Im\,\varphi_Z\left(u; \underline{s}_2\right)\right)^2 V_{Y,\Im}\left(u; \underline{s}_1\right) \\ &\qquad\quad + 2\left(\Re\,\varphi_Y\left(u; \underline{s}_1\right) \operatorname{Cov}\left(G_{X,\Re}\left(u; \underline{s}\right), G_{Z,\Re}\left(u; \underline{s}_2\right)\right)\right. \\ &\qquad\qquad - \Im\,\varphi_Y\left(u; \underline{s}_1\right) \operatorname{Cov}\left(G_{X,\Re}\left(u; \underline{s}\right), G_{Z,\Im}\left(u; \underline{s}_2\right)\right) \\ &\qquad\qquad + \Re\,\varphi_Z\left(u; \underline{s}_2\right) \operatorname{Cov}\left(G_{X,\Re}\left(u; \underline{s}\right), G_{Y,\Re}\left(u; \underline{s}_1\right)\right) \\ &\qquad\qquad - \Im\,\varphi_Z\left(u; \underline{s}_2\right) \operatorname{Cov}\left(G_{X,\Re}\left(u; \underline{s}\right), G_{Y,\Im}\left(u; \underline{s}_1\right)\right) \\ &\qquad - \Re\,\varphi_Y\left(u; \underline{s}_1\right) \Im\,\varphi_Y\left(u; \underline{s}_1\right) \operatorname{Cov}\left(G_{Z,\Re}\left(u; \underline{s}_2\right), G_{Z,\Im}\left(u; \underline{s}_2\right)\right) \\ &\qquad\qquad\qquad \left. - \Re\,\varphi_Z\left(u; \underline{s}_2\right) \Im\,\varphi_Z\left(u; \underline{s}_2\right) \operatorname{Cov}\left(G_{Y,\Re}\left(u; \underline{s}_1\right), G_{Y,\Im}\left(u; \underline{s}_1\right)\right)\right). \end{aligned}$$

Concerning the second variance in question, it holds

$$
\begin{aligned}
&\operatorname{Var}\left(G_{\Im}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\\
&= V_{X,\Im}\left(u;\underline{s}\right)+\left(\Im\,\varphi_Y\left(u;\underline{s}_1\right)\right)^2 V_{Z,\Re}\left(u;\underline{s}_2\right)+\left(\Re\,\varphi_Y\left(u;\underline{s}_1\right)\right)^2 V_{Z,\Im}\left(u;\underline{s}_2\right)\\
&\qquad+\left(\Im\,\varphi_Z\left(u;\underline{s}_2\right)\right)^2 V_{Y,\Re}\left(u;\underline{s}_1\right)+\left(\Re\,\varphi_Z\left(u;\underline{s}_2\right)\right)^2 V_{Y,\Im}\left(u;\underline{s}_1\right)\\
&\qquad+2\left(\Im\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Z,\Re}\left(u;\underline{s}_2\right)\right)\right.\\
&\qquad+\Re\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Z,\Im}\left(u;\underline{s}_2\right)\right)\\
&\qquad+\Im\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Y,\Re}\left(u;\underline{s}_1\right)\right)\\
&\qquad+\Re\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Y,\Im}\left(u;\underline{s}_1\right)\right)\\
&\qquad+\Im\,\varphi_Y\left(u;\underline{s}_1\right)\Re\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{Z,\Re}\left(u;\underline{s}_2\right),G_{Z,\Im}\left(u;\underline{s}_2\right)\right)\\
&\qquad\left.+\Im\,\varphi_Z\left(u;\underline{s}_2\right)\Re\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{Y,\Re}\left(u;\underline{s}_1\right),G_{Y,\Im}\left(u;\underline{s}_1\right)\right)\right).
\end{aligned}
$$

Joining these two leads to

$$
\begin{aligned}
&V_G\left(u;\underline{s}_1,\underline{s}_2\right)\\
&= V_{X,\Re}\left(u;\underline{s}\right)+V_{X,\Im}\left(u;\underline{s}\right)+\left|\varphi_Y\left(u;\underline{s}_1\right)\right|^2 V_{Z,\Re}\left(u;\underline{s}_2\right)+\left|\varphi_Y\left(u;\underline{s}_1\right)\right|^2 V_{Z,\Im}\left(u;\underline{s}_2\right)\\
&\qquad+\left|\varphi_Z\left(u;\underline{s}_2\right)\right|^2 V_{Y,\Re}\left(u;\underline{s}_1\right)+\left|\varphi_Z\left(u;\underline{s}_2\right)\right|^2 V_{Y,\Im}\left(u;\underline{s}_1\right)\\
&\qquad+2\left(\Re\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right),G_{Z,\Re}\left(u;\underline{s}_2\right)\right)\right.\\
&\qquad-\Im\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right),G_{Z,\Im}\left(u;\underline{s}_2\right)\right)\\
&\qquad+\Re\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right),G_{Y,\Re}\left(u;\underline{s}_1\right)\right)\\
&\qquad-\Im\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Re}\left(u;\underline{s}\right),G_{Y,\Im}\left(u;\underline{s}_1\right)\right)\\
&\qquad+\Im\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Z,\Re}\left(u;\underline{s}_2\right)\right)\\
&\qquad+\Re\,\varphi_Y\left(u;\underline{s}_1\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Z,\Im}\left(u;\underline{s}_2\right)\right)\\
&\qquad+\Im\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Y,\Re}\left(u;\underline{s}_1\right)\right)\\
&\qquad\left.+\Re\,\varphi_Z\left(u;\underline{s}_2\right)\operatorname{Cov}\left(G_{X,\Im}\left(u;\underline{s}\right),G_{Y,\Im}\left(u;\underline{s}_1\right)\right)\right).
\end{aligned}
\tag{A.271}
$$

Coming back to (A.270), bounding the expectation is equal to bounding $V_G\left(u;\underline{s}_1,\underline{s}_2\right)$. Using the Cauchy-Schwarz inequality, we are able to bound the covariances of (A.271) by variances, which appear in the said equation too. Thus, we only need to establish bounds for these variances. But before doing so, we use the fact that characteristic functions can be bounded by 1 and obtain

$$
\begin{aligned}
&V_G\left(u;\underline{s}_1,\underline{s}_2\right)\\
&\leq V_{X,\Re}\left(u;\underline{s}\right)+V_{X,\Im}\left(u;\underline{s}\right)+V_{Z,\Re}\left(u;\underline{s}_2\right)+V_{Z,\Im}\left(u;\underline{s}_2\right)+V_{Y,\Re}\left(u;\underline{s}_1\right)+V_{Y,\Im}\left(u;\underline{s}_1\right)\\
&\quad+2\Big(\left(V_{X,\Re}\left(u;\underline{s}\right)V_{Z,\Re}\left(u;\underline{s}_2\right)\right)^{1/2}+\left(V_{X,\Re}\left(u;\underline{s}\right)V_{Z,\Im}\left(u;\underline{s}_2\right)\right)^{1/2}+\left(V_{X,\Re}\left(u;\underline{s}\right)V_{Y,\Re}\left(u;\underline{s}_1\right)\right)^{1/2}\\
&\qquad+\left(V_{X,\Re}\left(u;\underline{s}\right)V_{Y,\Im}\left(u;\underline{s}_1\right)\right)^{1/2}\left(V_{X,\Im}\left(u;\underline{s}\right)V_{Z,\Re}\left(u;\underline{s}_2\right)\right)^{1/2}+\left(V_{X,\Im}\left(u;\underline{s}\right)V_{Z,\Im}\left(u;\underline{s}_2\right)\right)^{1/2}\\
&\qquad+\left(V_{X,\Im}\left(u;\underline{s}\right)V_{Y,\Re}\left(u;\underline{s}_1\right)\right)^{1/2}+\left(V_{X,\Im}\left(u;\underline{s}\right)V_{Y,\Im}\left(u;\underline{s}_1\right)\right)^{1/2}\Big).
\end{aligned}
\tag{A.272}
$$

At this point, we continue with the variances starting with $V_{X,\Re}\left(u;\underline{s}\right)$ and $V_{X,\Im}\left(u;\underline{s}\right)$. According to Lemma 5.2, it holds

$$V_{X,\Re}(u;\underline{s}) = \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right)\right)$$

and

$$V_{X,\Im}(u;\underline{s}) = \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \mathrm{Cov}\left(\sin\left(\langle \underline{s}, \underline{\widetilde{X}}_h(u)\rangle\right), \sin\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right)\right).$$

Since the integral is finite, the next step will be to bound the covariances in the equations above in such a way that they do not loose their ability to be totaled. Here, Lemma 5.9 combined with equation (A.258) comes in handy, and we can bound both $V_{X,\Re}(u;\underline{s})$ and $V_{X,\Im}(u;\underline{s})$ by

$$C\left(|\underline{s}|_1 \sum_{h\in\mathbb{N}} h^{-(1+\tilde{\delta})} + 1\right) = C_X\left(|\underline{s}|_1 + 1\right) \leq C_X\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1\right)$$

for some finite constant $C_X > 0$ using (A.267). In the same manner, we obtain $C_Y\left(|\underline{s}_1|_1 + 1\right)$ as an upper bound for $V_{Y,\Re}(u;\underline{s}_1)$ and $V_{Y,\Im}(u;\underline{s}_1)$. Similarly, $C_Z\left(|\underline{s}_2|_1 + 1\right)$ bounds both $V_{Z,\Re}(u;\underline{s}_2)$ and $V_{Z,\Im}(u;\underline{s}_2)$ from above. Here, C_Y and C_Z are both finite positive constants. Thus, we can return to equation (A.272) and bound $V_G(u;\underline{s}_1,\underline{s}_2)$ by

$$\begin{aligned}
&C\Big(\left(|\underline{s}_1|_1 + 1\right) + \left(|\underline{s}_2|_1 + 1\right) + \left(\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1\right)\left(|\underline{s}_2|_1 + 1\right)\right)^{1/2} \\
&\qquad\qquad + \left(\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1\right)\left(|\underline{s}_1|_1 + 1\right)\right)^{1/2}\Big) \\
&\leq C\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + \left(|\underline{s}_1|_1\,|\underline{s}_2|_1 + 1\right)^{1/2}\right) \\
&\leq C\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + \left(\max\left\{|\underline{s}_1|^2, |\underline{s}_2|^2\right\}\right)^{1/2} + 1\right) \\
&= C\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + \max\left\{|\underline{s}_1|_1, |\underline{s}_2|_1\right\} + 1\right) \\
&=: C_{V_G}\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1\right)
\end{aligned} \tag{A.273}$$

with some finite constant $C_{V_G} > 0$. Knotting it with equation (A.270), the bound established above functions a bound for the integrand of (A.269) as well. This leads to

$$E\,\mathcal{G}_\eta = C_{V_G} \int_{D_\eta} \left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1\right) d\mathfrak{w} < \infty$$

with the help of Assumption 10. Using the same arguments, it follows $E\mathcal{G} < \infty$. Thus, considering the $\mathcal{L}_1$-distance we obtain

$$E\left|\mathcal{G}_\eta - \mathcal{G}\right| = \int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} E\left|G\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2 d\mathfrak{w} \longrightarrow 0$$

as $\eta \to 0$ due to bound of $E\left|G\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2$ established in (A.273) and our choice of the weight function. Hence, $\mathcal{G}_\eta$ converges in the first mean to $\mathcal{G}$, and therefore convergence in distribution follows immediately. This finishes constraint (2) and requirement (3) is left to look at. With the use of Markov's inequality and, again, Fubini's theorem, we get

$$
\begin{aligned}
&P\left(\left|b_T T \int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} \left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w}\right|>\epsilon\right)\\
&=P\left(\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} b_T T\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w}>\epsilon\right)\\
&\le \epsilon^{-1}E\left(\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} b_T T\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w}\right)\\
&=\epsilon^{-1}\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} b_T TE\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w}. \qquad \text{(A.274)}
\end{aligned}
$$

The integrand from above can be rewritten and bounded as seen in (A.263) in order to obtain

$$
\begin{aligned}
&b_T TE\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2\\
&\le b_T TE\left(\left(\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|\right.\right.\\
&\qquad\left.\left.+\left|\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|+\left|\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right|\right)^2\right). \qquad \text{(A.275)}
\end{aligned}
$$

Comparable to equation (A.264), we continue with

$$
\begin{aligned}
&b_T TE\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2+b_T TE\left|\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2\\
&\qquad+b_T TE\left|\varphi_Y\left(u;\underline{s}_1\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\right|^2\\
&=: \mathrm{I}+\mathrm{II}+\mathrm{III} \qquad \text{(A.276)}
\end{aligned}
$$

again disregarding any occurring constants.

Now we use the bounds established in the proof of Theorem 5.13 to bound the summands of (A.275). For term I, it holds

$$
\begin{aligned}
\mathrm{I}&=b_T T\left(\left|\underline{s}\right|_1^{\frac{2+\delta}{2}}\mathcal{O}\left(b_T^{\frac{2+\delta}{2}}\right)+\left(\left|\underline{s}\right|_1+1\right)\mathcal{O}\left(\left(b_T T\right)^{-1}\right)\right)\\
&=\left(\left|\underline{s}\right|_1^{\frac{2+\delta}{2}}+1\right)\mathcal{O}(1)\\
&=\left(\left|\underline{s}_1\right|_1^{\frac{2+\delta}{2}}+\left|\underline{s}_2\right|_1^{\frac{2+\delta}{2}}+1\right)\mathcal{O}(1)
\end{aligned}
$$

due to Assumption 9 and $b_T^{\frac{4+\delta}{2}}T=\mathcal{O}(1)$. Analogously, we have

$$
\mathrm{II}=\left(\left|\underline{s}_2\right|_1^{\frac{2+\delta}{2}}+1\right)\mathcal{O}(1) \qquad \text{and} \qquad \mathrm{III}=\left(\left|\underline{s}_1\right|_1^{\frac{2+\delta}{2}}+1\right)\mathcal{O}(1).
$$

Consequently, we can bound the integrand in (A.274) by

$$
\mathcal{O}(1)\left(\left|\underline{s}_1\right|_1^{\frac{2+\delta}{2}}+\left|\underline{s}_2\right|_1^{\frac{2+\delta}{2}}+1\right).
$$

Again, there would be no further use in exploiting the independence of $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$. Returning to (A.274), by inserting the upper bound said equation becomes

$$\mathcal{O}(1)\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta}\left(|\underline{s}_1|_1^{\frac{2+\delta}{2}}+|\underline{s}_2|_1^{\frac{2+\delta}{2}}+1\right)d\mathfrak{w},$$

which tends to 0 as η tends to 0 due to Assumption 10. This closes the proof. □

The last proof of this section deals with the dependent counterpart of the previously demonstrated theorem.

Proof of Lemma 5.16. In order to make use of Theorem 5.6, consider

$$D_\eta := \left\{(\underline{s}_1',\underline{s}_1')'\,\middle|\,\eta\le|\underline{s}_1|_2\,,|\underline{s}_2|_2\le 1/\eta\right\}$$

again for $\eta\in(0,1)$. Therewith, we divide the integral the following way:

$$\begin{aligned}
&b_T T\,\widehat{\mathfrak{C}}_{Y,Z}(u)\\
&=b_T T\int_{D_\eta}\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w}\\
&\qquad\qquad+b_T T\int_{\{\mathbb{R}^p\times\mathbb{R}^q\}\setminus D_\eta}\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w}.
\end{aligned}$$

Due to the non-negative weight function w, the integrand of $b_T T\,\widehat{\mathfrak{C}}_{Y,Z}(u)$ is non-negative as well. This leads to

$$b_T T\,\widehat{\mathfrak{C}}_{Y,Z}(u)\ge b_T T\int_{D_\eta}\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2 d\mathfrak{w}. \tag{A.277}$$

Thus, it suffice to show the divergence in probability of the right-hand side of (A.277). Therefore, we limit ourselves to the examination of said term. To begin with, we focus on the integrand. Comparable to the proof of Theorem 5.14, we insert the characteristic function of the joint random vector consisting of $\left(\underline{Y}_{t,T}\right)$ and $\left(\underline{Z}_{t,T}\right)$. In doing so, we obtain

$$\begin{aligned}
&\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2\\
&\quad=\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)+\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2.
\end{aligned}\tag{A.278}$$

In opposition to the previous theorem, $\underline{\widetilde{Y}}_0(u)$ and $\underline{\widetilde{Z}}_0(u)$ are claimed to be dependent. Therefore, we need to infix the individual characteristic functions of both said random variables directly without the possibility of transforming the joint one. This leads to

$$\begin{aligned}
&\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)+\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right)\right.\\
&\qquad\qquad\left.+\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right)-\widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2
\end{aligned}$$

$$= \left|\left(\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) + \varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right) - \left(\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\right|^2 \tag{A.279}$$

instead of equation (A.278) to move on with. Using the reverse triangle inequality gives

$$\left|\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) + \varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right| - \left|\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|\right|^2 \tag{A.280}$$

as a lower bound for (A.279). Therefore, we continue with (A.280). Because of the squaring, we can neglect the outer absolute value. Subsequently, we perform a binomial expansion and obtain

$$\begin{aligned}&\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) + \varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|^2\\&\quad - 2\left|\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) + \varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right|\\&\quad\quad \cdot\left|\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|\\&\quad\quad\quad + \left|\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2.\end{aligned} \tag{A.281}$$

At this point, we take the prefactor $b_T T$ back into account. This leads to

$$\begin{aligned}&\left|\left(b_T T\right)^{1/2}\left(\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) + \varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\right|^2\\&\quad - 2\left|\left(b_T T\right)^{1/2}\left(\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right.\right.\\&\quad\quad \left.\left.+\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\right|\\&\quad\quad \cdot\left(b_T T\right)^{1/2}\left|\left(\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\right|\\&\quad\quad\quad + b_T T\left|\left(\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\right|^2,\end{aligned}$$

which can be bounded from beneath by

$$\begin{aligned}&\left(b_T T\right)^{1/2}\left|\left(\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\right|\\&\quad \cdot\Big(\left(b_T T\right)^{1/2}\left|\left(\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\right|\\&\quad\quad - 2\left|\left(b_T T\right)^{1/2}\left(\widehat{\varphi}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - \varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right.\right.\\&\quad\quad\quad \left.\left.+\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right) - \widehat{\varphi}_Y\left(u;\underline{s}_1\right)\widehat{\varphi}_Z\left(u;\underline{s}_2\right)\right)\right|\Big).\end{aligned} \tag{A.282}$$

The subtrahend in (A.282) can be treated as in the proof of Theorem 5.14 because at this point, the independence was not borne in mind. Thus, we obtain convergence in distribution to the absolute value of a Gaussian process due to the compact space D_η given by the integral by using Theorem 5.6. Therefore, we can equal this term with $\mathcal{O}_P(1)$. Consequently, our attention shifts towards the minuend, which is of the same form as the first factor of (A.282). Due to the underlying dependence of the processes, it holds

$$\left|\left(\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\right|>0.$$

Moreover, the left-hand side above is independent of T. Hence, we get

$$\left(b_T T\right)^{1/2}\left|\left(\varphi_Y\left(u;\underline{s}_1\right)\varphi_Z\left(u;\underline{s}_2\right)-\varphi_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right)\right|=C\left(b_T T\right)^{1/2}$$

which tends to ∞ while $T\to\infty$. In conclusion, by combining all bounds belonging to the components of equation (A.282), we obtain divergence in probability of the lower bound of the initial integrand diverges. Because this behavior translates to the integrand in the first place and thus to the integral itself, the proof is finished. □

The last proof terminates this section and we pursue with its bootstrap analogue in the next section.

A.5. Proofs Belonging to Chapter 6

In this section, we are going to prove the bootstrap counterparts of the results presented in Chapter 5. This is the reason why the structure of the following subsections stays the same as in the previous section. Nevertheless, as we are back in the bootstrap world, some additional results will be found in this section, which are needed to transfer the proofs, for example the determination of suitable good sets.

A.5.1. Proofs of Section 6.2

In this subsection, we work our way up to the proof of of the bootstrap consistency theorem. On this way, we will encounter proofs dealing with the ECF-case as well as general ones. Moreover, we will also find some additional auxiliary result and their proofs.

However, the first result we are going to demonstrate is the one belonging to Lemma 6.2 and concentrates on the ECF case:

Proof of Lemma 6.2. To show the wanted results, we take either the P-expectation or both the P-expectation and the $P^\star$-expectation into account as the searched-after rates hold in P-probability. This approach is justified by Markov's inequality. The involvement of the bootstrap expectation in the first case is reasoned by the sought-after $\mathcal{O}_{P^\star}$-term. The proof orientates itself mostly by the proof of Lemma 5.11. The main difference is the relinquishment of the use of Jensen's inequality. This is justified by the fact that we do not need to maximize the exponent belonging to the bandwidth b_T, as there is no T to be balanced out in further use of the result. Thus, the upper bound is a little less sharp, but the proof becomes shorter and less complex. While the proof of the first part will rely heavily on the results shown in the proof of Lemma 5.11, the proof of the second part uses rather the structure. During the whole proof, we concentrate of non-endpoint cases because the neglected ones can be conducted in the very same way. The only difference lays in the index shifts, which do not have any impact in the end result as the sums will be canceled out due to universal upper bounds and the stationarity of the companion process.

(i) First, we have a closer look at $\widehat{\varphi}_X^\star(u;\underline{s})$. This expression consists of observations of the locally stationary process, whereas the companion counterpart is used in $\varphi_X(u;\underline{s})$. Thus, in order to handle the difference in question we need to build a bridge between these two terms. This bridge consists of inserting a bootstrap analogue of (5.5) while taking the bootstrap expectation into account. In other words, we add and subtract a weighted sum consisting of $\widehat{\widetilde{\varphi}}(u;\underline{s})$ with a suitably shifted index. This leads to

$$\begin{aligned}
&E^\star\left|\widehat{\varphi}_X^\star(u;\underline{s})-\varphi_X(u;\underline{s})\right|^2\\
&=\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{X}_{t+r,T}\rangle}\right.\\
&\qquad\qquad-\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}\\
&\qquad\qquad\left.+\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}-\varphi_X(u;\underline{s})\right|^2. \qquad (A.283)
\end{aligned}$$

By expansion of the product above, we obtain the following upper bound for (A.283) to work with regardless any constants:

$$\begin{aligned}
&\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)\left(e^{i\langle\underline{s},\underline{X}_{t+r,T}\rangle}-e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}\right)\right|^2\\
&\qquad+\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}-\varphi_X(u;\underline{s})\right|^2\\
&=:\mathrm{I}+\mathrm{II}. \qquad (A.284)
\end{aligned}$$

We start with the second newly-defined term. Since we want the $\mathcal{O}_{P^\star}$-terms to hold in P-probability, we take the real world expectation into account. That is why we consider

$$E\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}-\varphi_X(u;\underline{s})\right|^2\right).$$

As the companion process is stationary in u, the sum over r is set off by the prefactor, and we can directly take the results from the proof of Lemma 5.11 over, that is

$$\mathrm{II} = \left(|\underline{s}|_1 + 1\right) \mathcal{O}_{P^\star}\left(\frac{1}{b_T T}\right)$$

in P-probability. Therefore, we move on to the first term of equation A.284 and begin with the decomposition of the absolute value. This leads to

$$\begin{aligned}
\mathrm{I} = &\frac{1}{2TD_T + 1} \sum_{r=-TD_T}^{TD_T} \left(\Re \left(\frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) \left(e^{i\langle \underline{s}, \underline{X}_{t+r,T}\rangle} - e^{i\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle} \right)\right)\right)^2 \\
&+ \frac{1}{2TD_T + 1} \sum_{r=-TD_T}^{TD_T} \left(\Im \left(\frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) \left(e^{i\langle \underline{s}, \underline{X}_{t+r,T}\rangle} - e^{i\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle} \right)\right)\right)^2 \\
=: &\ \mathrm{Ia} + \mathrm{Ib}.
\end{aligned} \tag{A.285}$$

Because considering the real part means working with the cosine and looking at the imaginary part implies dealing with the sine, it suffices to examine only one of them in detail. In our case, this will be the real part, to wit subterm Ia. By expanding the square, we obtain

$$\begin{aligned}
\mathrm{Ia} = &\frac{1}{(b_T T)^2} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} K\left(\frac{t_1/T - u}{b_T}\right) K\left(\frac{t_2/T - u}{b_T}\right) \\
&\cdot \frac{1}{2TD_T + 1} \sum_{r=-TD_T}^{TD_T} \left(\left(\cos\left(\langle \underline{s}, \underline{X}_{t_1+r,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\rangle\right)\right)\right. \\
&\qquad \left.\cdot \left(\cos\left(\langle \underline{s}, \underline{X}_{t_2+r,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\rangle\right)\right)\right).
\end{aligned} \tag{A.286}$$

Since the cosine function in bounded by 1, we can neglect the second factor in (A.286) by replacing it with a constant. This implies the vanishing of one of the sums including the prefactor and the belonging kernel function. By additionally taking the absolute value, this results in

$$\frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T - u}{b_T}\right) \frac{1}{2TD_T + 1} \sum_{r=-TD_T}^{TD_T} \left| \cos\left(\langle \underline{s}, \underline{X}_{t+r,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right)\right|. \tag{A.287}$$

As this bounds equation (A.286), we can go on with the easier term (A.287). Because we want to benefit from the closeness between the locally stationary process and the companion one again, we use the Lipschitz condition (5.3) once more to get for every $r \in [-TD_T, TD_T]$

$$\left|\cos\left(\langle \underline{s}, \underline{X}_{t+r,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right)\right| \leq |\underline{s}|_1 \left|\underline{X}_{t+r,T} - \underline{\widetilde{X}}_{t+r}(u)\right|_1 .$$

In order to utilize the results discussing the closeness between $\underline{X}_{t+r,T}$ and $\underline{\widetilde{X}}_{t+r}\left(\frac{t+r}{T}\right)$ just as between $\underline{\widetilde{X}}_{t+r}\left(\frac{t+r}{T}\right)$ and $\underline{\widetilde{X}}_{t+r}(u)$, we continue anew with the plug-in of the required version of the companion process into the right-hand side of (A.287). Hence, we obtain

$$\begin{aligned}
&|\underline{s}|_1 \left|\underline{X}_{t+r,T} - \underline{\widetilde{X}}_{t+r}(u)\right|_1 \\
&\qquad \leq |\underline{s}|_1 \left(\left|\underline{X}_{t+r,T} - \underline{\widetilde{X}}_{t+r}\left(\frac{t+r}{T}\right)\right|_1 + \left|\underline{\widetilde{X}}_{t+r}\left(\frac{t+r}{T}\right) - \underline{\widetilde{X}}_{t+r}(u)\right|_1\right).
\end{aligned}$$

Now the groundwork is done, which permits the application of Lemma 2.3 to declare the bound stated below:

$$\begin{aligned}
\left|\cos\left(\langle \underline{s}, \underline{X}_{t+r,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right)\right| &= |\underline{s}|_1 \left(\mathcal{O}_P\left(T^{-1}\right) + \mathcal{O}_P\left(\left|\frac{t+r}{T} - u\right|\right)\right) \\
&= |\underline{s}|_1 \, \mathcal{O}_P\left(\frac{1}{T} + \left|\frac{t}{T} - u\right| + \left|\frac{r}{T}\right|\right) \\
&= |\underline{s}|_1 \, \mathcal{O}_P\left(b_T + D_T\right). \qquad \text{(A.288)}
\end{aligned}$$

Note that the last step has its justification in the fact that we are in a situation where the absolute value of the kernel's argument is smaller or equal to 1. Otherwise, the kernel is always zero. Because only the index t without the additional r is responsible for the kernel's value, this means the argument satisfies $\left|\frac{t}{T} - u\right| \leq b_T$. The vanishing of T^{-1} is due to the fact that it is smaller than b_T for sufficiently large T by Assumption 9. At the same time, it holds $b_T > T^{-1/2}$ for T large enough due to the same assumption. However, Assumption 6 gives $D_T \leq d_T^{-\frac{1}{2+\delta}} T^{-1}$ and thus $D_T \leq T^{-\frac{1+\delta}{2+\delta}}$ as well since the number of non-zero weights d_T^{-1} is naturally at most equal to T. In this context, we claim

$$D_T \leq b_T \qquad \text{(A.289)}$$

for T large enough. In order to verify this assertion, it suffices to compare the exponents of T appearing in the lower bound of b_T and in the upper bound of D_T, respectively. Because

$$\frac{1+\delta}{2+\delta} - \frac{1}{2} > 0$$

is equivalent to $\delta > 0$, the proposition in equation (A.289) is true for all choices of $\delta \in (0,1)$ while T is sufficiently large. Therefore, we can simplify the upper bound in equation (A.288) and have in this way

$$\left|\cos\left(\langle \underline{s}, \underline{X}_{t+r,T}\rangle\right) - \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right)\right| = |\underline{s}|_1 \mathcal{O}_P(b_T)$$

uniformly for all $r \in [-TD_T, TD_T]$. Consequently, the sum over r in Ia cancels out, and we obtain

$$\text{Ia} \leq \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)|\underline{s}|_1 \mathcal{O}_P(b_T) = |\underline{s}|_1 \mathcal{O}_P(b_T)$$

in lieu of equation (A.287). As explained earlier, we can transfer this bound to subterm Ib of equation (A.285). This results in $\text{I} = |\underline{s}|_1 \mathcal{O}_P(b_T)$ as well. Combined with the rate we already established for term II of (A.284) Consequently, it holds

$$E^{\star}\left|\widehat{\varphi}^{\star}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right|^2 = |\underline{s}|_1 \mathcal{O}(b_T) + (|\underline{s}|_1+1)\mathcal{O}\left((b_T T)^{-1}\right) \tag{A.290}$$

in P-probability. Lastly, we want to slim down the right-hand side of equation (A.290). To begin with, it holds $|\underline{s}|_1 < |\underline{s}|_1 + 1$. Then we take a closer look at the $\mathcal{O}$-terms. We state

$$\mathcal{O}\left((b_T T)^{-1}\right) \subseteq \mathcal{O}(b_T). \tag{A.291}$$

To see this, we show the positivity of the difference of the arguments in the right way, that is

$$b_T - \frac{1}{b_T T} > 0.$$

This is equivalent to $b_T^2 T > 1$, which holds true for adequately large T by Assumption 9. In consequence, (A.291) is verified. Therefore, we can replace equation (A.290) by

$$\left|\widehat{\varphi}^{\star}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right|^2 = (|\underline{s}|_1+1)\mathcal{O}_{P^{\star}}(b_T)$$

in P-probability, which ends the first part of the proof.

(ii) Similar to the previous part, we start by inserting some kind of bootstrap expectation analogue of (5.5). This leads to

$$\begin{aligned} &E\left|E^{\star}\widehat{\varphi}^{\star}_X(u;\underline{s}) - \varphi_X(u;\underline{s})\right|^2 \\ &= E\left|E^{\star}\widehat{\varphi}^{\star}_X(u;\underline{s}) - \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle}\right. \\ &\qquad \left. + \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle} - \varphi_X(u;\underline{s})\right|^2. \end{aligned} \tag{A.292}$$

Again, we expand the product to obtain an upper bound for (A.292). By disregarding any upcoming constants, we continue our calculations with

$$\begin{aligned}
&E\left|E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}\right|^2\\
&\qquad+E\left|\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}-\varphi_X\left(u;\underline{s}\right)\right|^2\\
&=:\mathrm{I}+\mathrm{II}
\end{aligned}\tag{A.293}$$

instead of equation (A.292). Starting with the first, we examine the arisen terms I and II singly. In the beginning, we convert the squared absolute value into a sum of squared real and imaginary part, namely

$$\begin{aligned}
\mathrm{I}&=E\left(\Re\left(E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}\right)\right)^2\\
&\quad+E\left(\Im\left(E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}\right)\right)^2\\
&=:\mathrm{Ia}+\mathrm{Ib}.
\end{aligned}\tag{A.294}$$

Following the same argumentation as in part (i), we limit ourselves to the examination of Ia. Both minuend and subtrahend consist of a sum over r and one over t. Thus, we pool those sums and expand the square to obtain

$$\begin{aligned}
\mathrm{Ia}=\frac{1}{\left(2TD_T+1\right)^2}&\sum_{r_1,r_2=-TD_T}^{TD_T}\frac{1}{\left(b_TT\right)^2}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}K\left(\frac{t_1/T-u}{b_T}\right)K\left(\frac{t_2/T-u}{b_T}\right)\\
&\cdot E\left(\left(\cos\left(\langle\underline{s},\underline{X}_{t_1+r_1,T}\rangle\right)-\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_1+r_1}(u)\rangle\right)\right)\right.\\
&\qquad\left.\cdot\left(\cos\left(\langle\underline{s},\underline{X}_{t_2+r_2,T}\rangle\right)-\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t_2+r_2}(u)\rangle\right)\right)\right).
\end{aligned}\tag{A.295}$$

Borrowing once again the arguments from part (i), we bound equation (A.295) by

$$\begin{aligned}
&\frac{C}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_TT}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)\\
&\qquad\cdot E\left|\cos\left(\langle\underline{s},\underline{X}_{t+r,T}\rangle\right)-\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle\right)\right|.
\end{aligned}\tag{A.296}$$

At this point, we remember the calculations made in the previous part and obtain

$$\mathrm{Ia} = \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t_1/T-u}{b_T}\right)|\underline{s}|_1\,\mathcal{O}\left(b_T\right) = |\underline{s}|_1\,\mathcal{O}\left(b_T\right)$$

in lieu of equation (A.295). The very same holds true for subterm Ib of A.294, which gives us $\mathrm{I} = |\underline{s}|_1\,\mathcal{O}\left(b_T\right)$ as result. Now we move on to term II of (A.293). As before, we start by decomposing the squared absolute value and get

$$\begin{aligned}\mathrm{II} = {} & E\left(\Re\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}-\varphi_X\left(u;\underline{s}\right)\right)\right)^2\\ & + E\left(\Im\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)e^{i\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle}-\varphi_X\left(u;\underline{s}\right)\right)\right)^2\\ & =: \mathrm{IIa}+\mathrm{IIb}. \end{aligned} \tag{A.297}$$

Again, we choose the first subterm, that is IIa, to have a closer look at. Comparable to the proof of Lemma 5.11, we rewrite said term to obtain useful differences, to wit

$$\begin{aligned}\mathrm{IIa} = E\Bigg| & \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)\\ & \cdot\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle\right)-\Re\,\varphi_X\left(u;\underline{s}\right)\right)\\ & -\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)-1\right)\Re\,\varphi_X\left(u;\underline{s}\right)\Bigg|^2.\end{aligned}$$

An intuitive bound for this expression is given by

$$\begin{aligned} E\Bigg(\Bigg| & \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle\right)-\Re\,\varphi_X\left(u;\underline{s}\right)\right)\Bigg|\\ & +\left|\left(\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)-1\right)\Re\,\varphi_X\left(u;\underline{s}\right)\right|\Bigg)^2,\end{aligned}$$

which, in turn, can be bounded by

$$\begin{aligned} E\Bigg| & \frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(\langle\underline{s},\underline{\widetilde{X}}_{t+r}(u)\rangle\right)-\Re\,\varphi_X\left(u;\underline{s}\right)\right)\Bigg|^2\\ & +\left(\frac{1}{b_T T}\sum_{t=1}^{T}K\left(\frac{t/T-u}{b_T}\right)-1\right)^2\end{aligned}$$

$$=: \mathrm{IIaa} + \mathrm{IIab} \tag{A.298}$$

by wilful neglect of the arising constants. We continue with a differentiated inspection of the subterms IIa and IIb starting with the latter. With the help of Lemma 4.2, we have

$$\mathrm{IIab} = \mathcal{O}\left(\frac{1}{(b_T T)^2}\right).$$

Next, we look at subterm IIaa of (A.298). Taking the stationarity of the companion process into account, we obtain

$$\begin{aligned}
\mathrm{IIaa} &= E\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\right.\\
&\qquad\left.\cdot\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right) - E\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right)\right)\right)^2\\
&\leq \frac{1}{(2TD_T+1)^2}\sum_{r_1,r_2=-TD_T}^{TD_T}\frac{C}{(b_T T)^2}\sum_{t_1,t_2=1}^{2\lfloor b_T T\rfloor+1}\\
&\qquad E\left(\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1+r_1}(u)\rangle\right) - E\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1+r_1}(u)\rangle\right)\right)\right.\\
&\qquad\left.\cdot\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2+r_2}(u)\rangle\right) - E\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2+r_2}(u)\rangle\right)\right)\right)\\
&= \frac{1}{(2TD_T+1)^2}\sum_{r_1,r_2=-TD_T}^{TD_T}\frac{C}{(b_T T)^2}\sum_{t_1,t_2=1}^{2\lfloor b_T T\rfloor+1}\\
&\qquad \mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1+r_1}(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_2+r_2}(u)\rangle\right)\right)\\
&= \frac{1}{(2TD_T+1)^2}\sum_{r_1,r_2=-TD_T}^{TD_T}\frac{C}{(b_T T)^2}\sum_{t_1,t_2=1}^{2\lfloor b_T T\rfloor+1}\\
&\qquad \mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t_1-t_2+r_1-r_2}(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{0}(u)\rangle\right)\right)\\
&= \frac{1}{(2TD_T+1)^2}\sum_{r=-2TD_T}^{2TD_T}\left(2TD_T+1-|r|\right)\frac{C}{(b_T T)^2}\\
&\qquad \cdot\sum_{t=-2\lfloor b_T T\rfloor}^{2\lfloor b_T T\rfloor}\left(\lfloor b_T T\rfloor+1-|t|\right)\mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{0}(u)\rangle\right)\right)
\end{aligned}$$

$$= \frac{1}{\left(2\,TD_T+1\right)^2} \sum_{r=-2\,TD_T}^{2\,TD_T} \left(2\,TD_T+1-|r|\right) \frac{C}{\left(b_T T\right)^2}$$
$$\cdot \left(\sum_{\substack{t=-2\lfloor b_T T\rfloor \\ t\neq -r}}^{2\lfloor b_T T\rfloor} \left(2\left\lfloor b_T T\right\rfloor+1-|t|\right) \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{t+r}(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{0}(u)\rangle\right)\right)\right.$$
$$\left. + \left(2\left\lfloor b_T T\right\rfloor+1-|r|\right) \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{0}(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{0}(u)\rangle\right)\right)\right). \tag{A.299}$$

Under consideration of equation (A.258) and Lemma 5.9, we bound (A.299) by

$$\frac{1}{\left(2\,TD_T+1\right)^2} \sum_{r=-2\,TD_T}^{2\,TD_T} \left(2\,TD_T+1-|r|\right) \frac{C}{\left(b_T T\right)^2}$$
$$\cdot \left(\sum_{\substack{t=-2\lfloor b_T T\rfloor \\ t\neq -r}}^{\lfloor b_T T\rfloor} \left(2\left\lfloor b_T T\right\rfloor+1-|t|\right) |\underline{s}|_1 \frac{C_{Cov,cs}}{|t+r|^{1+\tilde{\delta}}} + 4\left(2\left\lfloor b_T T\right\rfloor+1-|r|\right)\right)$$
$$\leq \frac{1}{\left(2\,TD_T+1\right)^2} \sum_{r=-2\,TD_T}^{2\,TD_T} \left(2\,TD_T+1-|r|\right) \frac{C}{\left(b_T T\right)^2}$$
$$\cdot \left(\sum_{\substack{t=-2\lfloor b_T T\rfloor+r \\ t\neq 0}}^{\lfloor b_T T\rfloor+r} \left(2\left\lfloor b_T T\right\rfloor+1\right) |\underline{s}|_1 \frac{C_{Cov,cs}}{|t|^{1+\tilde{\delta}}} + 4\left(2\left\lfloor b_T T\right\rfloor+1\right)\right)$$
$$\leq \frac{1}{\left(2\,TD_T+1\right)^2} \sum_{r=-2\,TD_T}^{2\,TD_T} \left(2\,TD_T+1\right) \frac{C}{\left(b_T T\right)^2}$$
$$\cdot \left(2\sum_{t\in\mathbb{N}} \left(2\left\lfloor b_T T\right\rfloor+1\right) |\underline{s}|_1 \frac{C_{Cov,cs}}{|t|^{1+\tilde{\delta}}} + 4\left(2\left\lfloor b_T T\right\rfloor+1\right)\right)$$
$$= \left(|\underline{s}|_1+1\right) \mathcal{O}\left(\frac{1}{b_T T}\right).$$

Thus, the combination of both IIaa and IIab results in

$$\mathrm{IIa} = \left(|\underline{s}|_1+1\right) \mathcal{O}\left(\frac{1}{b_T T}\right) + \mathcal{O}\left(\frac{1}{\left(b_T T\right)^2}\right) = \left(|\underline{s}|_1+1\right) \mathcal{O}\left(\frac{1}{b_T T}\right).$$

The very same steps lead to $\mathrm{IIb} = \left(|\underline{s}|_1+1\right) \mathcal{O}\left(\frac{1}{b_T T}\right)$ as well since both subterms IIa and IIb only differ with respect to the used trigonometric function. In consequence, we obtain the same rate for II of equation (A.293). Consequently, it holds

$$E\left|E^\star \widehat{\varphi}_X^\star\left(u;\underline{s}\right) - \varphi_X\left(u;\underline{s}\right)\right|^2 = |\underline{s}|_1\, \mathcal{O}\left(b_T\right) + \left(|\underline{s}|_1+1\right) \mathcal{O}\left(\left(b_T T\right)^{-1}\right),$$

which can be turned into

$$E\left|E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-\varphi_X\left(u;\underline{s}\right)\right|^2=\left(\left|\underline{s}\right|_1+1\right)\mathcal{O}\left(b_T\right)$$

by performing the same steps as in the first part of the proof. This ends part (ii) and hence the whole proof.

□

For the next proof, which is the one belonging to Lemma 6.4, we stay in the ECF-scenario.

Proof of Lemma 6.4. At first, we expand the difference in question and consider

$$\frac{1}{\left(b_T T\right)^{1/2}}\left(b_T T\right)^{1/2}\left(\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)\right). \tag{A.300}$$

Waiving $\left(b_T T\right)^{-1/2}$ for now, similarly to Theorem 3.23 and Theorem 3.2 in Jentsch et al. (2020) we have

$$\left(b_T T\right)^{1/2}\begin{pmatrix}\Re\left(\widehat{\varphi}_Y^{\star}\left(u;\underline{s}\right)-E^{\star}\widehat{\varphi}_Y^{\star}\left(u;\underline{s}\right)\right)\\ \Im\left(\widehat{\varphi}_Y^{\star}\left(u;\underline{s}\right)-E^{\star}\widehat{\varphi}_Y^{\star}\left(u;\underline{s}\right)\right)\end{pmatrix}_{\underline{s}\in[-S,S]^d}\xrightarrow{d}\begin{pmatrix}G_{Y,\Re}\left(u;\underline{s}\right)\\ G_{Y,\Im}\left(u;\underline{s}\right)\end{pmatrix}_{\underline{s}\in[-S,S]^d}$$

in P-probability as $T\to\infty$. In this context, $G_{X,\Re}\left(u;\underline{s}\right)$ and $G_{X,\Im}\left(u;\underline{s}\right)$ are centered Gaussian processes with covariance functions $\sigma^2_{X,\Re}\left(u;\underline{s},\underline{s}^{\circ}\right)$ and $\sigma^2_{X,\Im}\left(u;\underline{s},\underline{s}^{\circ}\right)$, respectively, having their origins in Lemma 5.2. Because the bootstrap ECF is a continuous function, we obtain with the use of the continuous mapping theorem that

$$\left(b_T T\right)^{1/2}\left(\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)\right)_{\underline{s}\in[-S,S]^d}$$

converges in distribution to a complex centered Gaussian process with continuous sample path. At this point, we come back to the neglected factor in (A.300). Since $\left(b_T T\right)^{-1/2}$ vanishes with T tending to ∞, the application of Slutsky's theorem leads to

$$\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)-E^{\star}\widehat{\varphi}_X^{\star}\left(u;\underline{s}\right)\xrightarrow{d}0$$

uniformly in $\underline{s}$ in P-probability as T tends to ∞. This implies also stochastic convergence in the bootstrap sense in P-probability as the limit is a constant. □

After having conducted two ECF-related proofs, we turn our attention back to the general case at this point. The next proof breaks the first ground in that regard.

Proof of Lemma 6.5. The proof is akin to the one belonging to Lemma 2.6, but here, the Lipschitz condition (5.3) stated in Assumption 11 is used in lieu of (2.11), which deals with bounded $\underline{s}$. That is also the reason why we do not consider the supremum over all $\underline{s}$ in this case. □

The next proof deals with products of f as arguments of the covariance and reads as follows:

Proof of Lemma 6.6. The proof can be performed analogously to the one of Lemma 3.14. The only difference lays in the use of Lemma 5.8 instead of Lemma 2.9. □

Now it is time for some additional results, which are needed in the upcoming proofs. At it, we stay in the context of products of the function f but move on to weighted sums of them in the altered version of Lemma A.8, namely:

Lemma A.14.
Let Assumptions 6, 9 and 11 for $k = 1$ hold true. Then, we have for all $\underline{s} \in \mathbb{R}^d$, $t_1, t_2 \in \mathbb{Z}$ and $u \in [0,1]$

(i)

$$\frac{1}{(2TD_T+1)^2} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \left| E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right)\right| \le \left(|\underline{s}|_1 + 1\right) \frac{C_{sum,1,b'}}{TD_T},$$

(ii)

$$\frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u)\right)\right) \le \left(|\underline{s}|_1 + 1\right) \frac{C_{sum,2,b'}}{TD_T}$$

and

(iii)

$$\frac{1}{(2TD_T+1)^4} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \sum_{m=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+k}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+m}(u)\right)\right) \le \left(|\underline{s}|_1 + 1\right) \frac{C_{sum,3,b'}}{TD_T}).$$

Here, $C_{sum,1,b'}, C_{sum,2,b'}$ and $C_{sum,3,b'}$ are positive finite constants independent of $\underline{s}, t_1, t_2$ and u. Moreover, $\bar{f}$ is defined as in (2.15) and denotes the centered version of f.

Proof. To prove the three inequalities stated above, we use the proof of Lemma A.8 as a guideline. For the first part, we will follow again the lines of the proof of Lemma A.1, the real world counterpart to this lemma, but when it comes to the second and third part, we can abridge the proof by explaining these cases by part (i).

(i) W.l.o.g. assume $t_1 \geq t_2$ once more. Because of stationarity, we have

$$\sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \left| E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u) \right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u) \right) \right) \right|$$
$$= \sum_{\substack{l=-2TD_T \\ t \neq t_2 - t_1}}^{2TD_T} (2TD_T + 1 - |l|) \left| \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1-t_2+l}(u) \right), f\left(\underline{s}, \underline{\widetilde{X}}_0(u) \right) \right) \right|$$
$$+ (2TD_T + 1 - |t_2 - t_1|) \left| \mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_0(u) \right), f\left(\underline{s}, \underline{\widetilde{X}}_0(u) \right) \right) \right|$$
$$=: \mathrm{I} + \mathrm{II}$$

likely to the proof of Lemma A.1. Thus, it holds $\mathrm{II} \leq C_{sum,1,b',a} TD_T$ for some finite constant $C_{sum,1,b',a} > 0$, Moving on to term I, in the same manner as in the previously mentioned proof we obtain

$$\mathrm{I} \leq |\underline{s}|_1 \sum_{l=-2TD_T}^{2TD_T} (2TD_T + 1 - |l|) \frac{C_{Cov,cs}}{|t_1 - t_2 + l|^{1+\tilde{\delta}}} = C_{sum,1,b',b} |\underline{s}|_1 TD_T$$

for some positive constant $C_{sum,1,b',b} > 0$ using Lemma 5.9. In combination, we get

$$\frac{1}{(2TD_T + 1)^2} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \left| E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u) \right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u) \right) \right) \right|$$
$$= (|\underline{s}|_1 + 1) \frac{C_{sum,1,b'}}{TD_T}.$$

(ii) Anew, w.l.o.g. we suppose $t_1 \geq t_2$. Besides, we work with $v := t_1 - t_2$. Analogously to the proof of the second part of Lemma A.1, stationarity of the companion process yields

$$\left| \frac{1}{(2TD_T + 1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \sum_{k=-TD_T}^{TD_T} \right.$$
$$\left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u) \right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u) \right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u) \right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+k}(u) \right) \right) \right|$$

$$\leq \left| \frac{1}{(2TD_T+1)^3} \sum_{r=-TD_T}^{TD_T} \sum_{\substack{l=-TD_T \\ l\neq r}}^{TD_T} \sum_{\substack{k=-TD_T \\ k\neq r}}^{TD_T} \right.$$

$$\left. E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right|$$

$$+ \left| \frac{1}{(2TD_T+1)^3} \sum_{\substack{l,r=-TD_T \\ l=r}}^{TD_T} \sum_{\substack{k=-TD_T \\ k\neq r}}^{TD_T} E\left(\bar{f}^2\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{k}(u)\right)\right) \right|$$

$$+ \left| \frac{1}{(2TD_T+1)^3} \sum_{\substack{r,k=-TD_T \\ r=k}}^{TD_T} \sum_{l=-TD_T}^{TD_T} E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+r}(u)\right) \bar{f}^2\left(\underline{s}, \underline{\widetilde{X}}_{r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v+l}(u)\right)\right) \right|$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Due to the boundedness of the function f, terms II and III can be bounded by $\frac{C_{sum,2,b',a}}{TD_T}$ for some positive finite constant $C_{sum,2,b',a}$. Hence, term I is left. Following the same argumentation as in the proof of the second part of Lemma A.8, we exchange the use of Lemma 3.4 for the application of Lemma 6.6 in the proceedings of the proof of part (ii) of Lemma A.1 to move on. This leads to III $\leq \frac{C_{sum,2,b',b}}{TD_T}$ for some constant $0 < C_{sum,2,b',b} < \infty$. Altogehter, we get

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = (|\underline{s}_1| + 1)\, \mathcal{O}\left(\frac{1}{TD_T}\right),$$

and this part of the proof is completed.

(iii) The argumentation for the third part of this proof can be taken directly from part (iii) of the proof appurtenant to Lemma A.8. Thus, the whole proof is brought to completion.

□

The next auxiliary lemma changes the second argument of products consisting of the function f from the local stationary to the companion process and thus forms a modification of Lemma A.9:

Lemma A.15.
Under Assumptions 4 for $k = 1$ and 3, we have for all $t_1, t_2 \in \{1, \ldots, T\}$

$$f\left(\underline{s}, \underline{X}_{t_1,T}\right) f\left(\underline{s}, \underline{X}_{t_2,T}\right) = f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right) + |\underline{s}|_1\, \mathcal{O}_P\left(T^{-1}\right).$$

At this, the $\mathcal{O}_P$-term is influenced neither by t_1, t_2 nor by $\underline{s}$.

Proof. To show the above-stated result, we follow exactly the lines of the proof corresponding to Lemma A.9 with the help of Lemma 6.5 in place of Lemma 2.6. □

As in the lemma above, we continue to change the second argument of f, but in this case, the change lies in the argument of the companion process itself. This leads to the following result:

Lemma A.16.
Suppose Assumptions 3, 4 for $k = 1$ and Assumption 6 are fulfilled. Then, for $\underline{s} \in \mathbb{R}^d$, $h, k \in \{1, \ldots, T\}$ and $-TD_T \le r, l \le TD_T$ it holds

$$f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right)$$
$$= f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right) + |\underline{s}|_1 \, \mathcal{O}_P\left(D_T\right).$$

Here, the $\mathcal{O}_P$-term is independent of the choices of h, k, r, l and $\underline{s}$.

Proof. The proof can be conducted by performing the very same calculations as in the proof of Lemma A.10, but using the second Lipschitz condition (5.3) instead of the first, that is (2.11). □

Now we return to the result of Section 6.2. With the help of the ancillary lemmata above, we perform the proof of Lemma 6.8 as follows:

Proof of Lemma 6.8. The proof follows the lines of the proof belonging to Lemma 3.6 because this is the counterpart dealing with bounded $\underline{s}$ and unbounded f. Thus, we start by rewriting the left-hand side of (6.2) and obtain

$$\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left(\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \right) \right.$$
$$\left. \cdot \left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} f\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \right) \right)$$
$$= \frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left(\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \right) \right.$$
$$\left. \cdot \left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1} \sum_{l=-TD_T}^{TD_T} \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right) \right) \right) \tag{A.301}$$

as before. Again, $\bar{f}$ denotes the centered version of the function f. The next step consists of bringing the real world covariance into play. Therefore, we look at the second moment of the difference between said covariance and equation $(A.301)$, that is

$$E\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left(\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u)\right)\right)\right.\right.$$
$$\left.\cdot\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u)\right)\right)\right)$$
$$\left.-\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)^2. \quad \text{(A.302)}$$

Performing the same calculations as in the proof of Lemma 3.6, we see that (A.302) equals

$$\frac{1}{(2TD_T+1)^2}\sum_{r=-TD_T}^{TD_T}\sum_{l=-TD_T}^{TD_T}$$
$$E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u)\right)\right)$$
$$+\frac{1}{(2TD_T+1)^4}\sum_{r=-TD_T}^{TD_T}\sum_{l=-TD_T}^{TD_T}\sum_{k=-TD_T}^{TD_T}\sum_{m=-TD_T}^{TD_T}$$
$$E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+k}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+m}(u)\right)\right)$$
$$-\frac{2}{(2TD_T+1)^3}\sum_{r=-TD_T}^{TD_T}\sum_{l=-TD_T}^{TD_T}\sum_{k=-TD_T}^{TD_T}$$
$$E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+k}(u)\right)\right)$$
$$+\frac{2}{(2TD_T+1)^2}\sum_{r=-TD_T}^{TD_T}\sum_{l=-TD_T}^{TD_T}E\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u)\right)\right)$$
$$\cdot\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\right)$$
$$-\left(\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)^2$$
$$=: \underline{\mathrm{I}}+\underline{\mathrm{II}}-\underline{\mathrm{III}}+\underline{\mathrm{IV}}-\underline{\mathrm{V}}. \quad \text{(A.303)}$$

To establish upper bound for the three middle terms of (A.303), we use Lemma A.14 and obtain

$$\underline{\mathrm{II}}=\left(|\underline{s}|_1+1\right)\mathcal{O}\left(\frac{1}{TD_T}\right),\ \underline{\mathrm{III}}=\left(|\underline{s}|_1+1\right)\mathcal{O}\left(\frac{1}{TD_T}\right)\ \text{and}\ \underline{\mathrm{IV}}=\left(|\underline{s}|_1+1\right)\mathcal{O}\left(\frac{1}{TD_T}\right).$$

This leaves us with the difference between $\underline{\mathrm{I}}$ and $\underline{\mathrm{V}}$. Again, we set $v := t_1 - t_2$, and by using the stationarity of the companion process, we obtain

$$\begin{aligned}
&|\underline{\mathrm{I}} - \underline{\mathrm{V}}| \\
&= \Bigg| \frac{1}{(2TD_T+1)^2} \sum_{r=-TD_T}^{TD_T} \sum_{l=-TD_T}^{TD_T} \\
&\qquad E\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+r}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+l}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2+l}(u)\right)\right) \\
&\qquad\qquad - \left(\mathrm{Cov}\left(f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u)\right), f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u)\right)\right)\right)^2 \Bigg| \\
&\leq \frac{2}{(2TD_T+1)^2} \sum_{t=1}^{2TD_T} (2TD_T+1-t) \\
&\qquad \cdot \left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t+v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| + \mathcal{O}\left(\frac{1}{(TD_T)^2}\right)
\end{aligned} \tag{A.304}$$

as in the proof of Lemma 3.6. Once again, to examine the left-over weighted sum further we consider different values of v solely beginning with $v > 0$. In doing so, it holds

$$\begin{aligned}
&\frac{2}{(2TD_T+1)^2} \sum_{t=1}^{2TD_T} (2TD_T+1-t) \\
&\qquad \cdot \left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t+v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\leq \frac{2}{2TD_T+1} \sum_{t=1}^{v-1} \left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t+v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\quad + \frac{2}{2TD_T+1} \sum_{t=v+1}^{2TD_T} \left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t+v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\qquad + \frac{2}{2TD_T+1} \left|\mathrm{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{2v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{v}(u)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{A.305}$$

Since the covariance is finite, we have $\mathrm{III} = \mathcal{O}\left((TD_T)^{-1}\right)$. Thus, we move on to term II. Looking at the summation bounds, it holds $t > v > 0$, which implies $t + v > t > v > 0$. In this context, our choice for the truncation parameter in (2.13) is $M := \left\lceil \frac{t-v}{2} \right\rceil$. Thus, we obtain with the help of part (i) of Lemma 6.6

$$\mathrm{II} \leq \frac{C}{2TD_T+1} \sum_{t=v+1}^{2TD_T} \frac{|\underline{s}|_1}{(t-v)^{1+\tilde{\delta}}} \leq \frac{C}{2TD_T+1} \sum_{t=1}^{2TD_T-v} \frac{|\underline{s}|_1}{t^{1+\tilde{\delta}}} = |\underline{s}|_1 \, \mathcal{O}\left(\frac{1}{TD_T}\right).$$

At this point, term I of equation (A.305) is left. Following the summation bounds, we have $v > t > 0$ and hence $t + v > v > t > 0$. This is the reason why we use part (ii) of Lemma 6.6 this time. To this end, have to use two different truncation parameters, that is $M_1 := \lceil t/2 \rceil$ and $M_2 := \left\lceil \frac{v-t}{2} \right\rceil$. Then, it follows with the use of the second part of Lemma 6.6

$$\begin{aligned}
\mathrm{I} &= \frac{2}{2TD_T+1}\sum_{t=1}^{v/2}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\quad + \frac{2}{2TD_T+1}\sum_{t=v/2+1}^{v-1}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right| \\
&\leq \frac{C}{2TD_T+1}\sum_{t=1}^{v/2}\frac{|\underline{s}|_1}{t^{1+\tilde{\delta}}} + \frac{C}{2TD_T+1}\sum_{t=v/2+1}^{v-1}\frac{|\underline{s}|_1}{(v-t)^{1+\tilde{\delta}}} \\
&= |\underline{s}|_1\,\mathcal{O}\left(\frac{1}{TD_T}\right).
\end{aligned}$$

This closes the first case. As in the proof of Lemma 3.6, there is no need to look at the case $v < 0$ because we can use the results from the first one. Hence, we go on with $v = 0$. Then, the weighted sum in (A.304) is equal to

$$\begin{aligned}
&\frac{2}{(2TD_T+1)^2}\sum_{t=1}^{2TD_T}(2TD_T+1-t) \\
&\qquad\cdot\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u)\right)\right)\right|
\end{aligned}$$

leaving us with $M := \lceil t/2 \rceil$ as truncation parameter. Following the lines belonging to the first case leads to the same result. Consequently, it holds $\underline{\mathrm{I}} - \underline{\mathrm{V}} = |\underline{s}|_1\,\mathcal{O}\left((TD_T)^{-1}\right)$ and therefore

$$\begin{aligned}
&\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left(\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u)\right)\right)\right. \\
&\qquad\left.\cdot\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u)\right) - \frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u)\right)\right)\right) \\
&= \mathrm{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u)\right),f\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u)\right)\right) + \left(|\underline{s}|_1+1\right)\mathcal{O}_P\left(\frac{1}{TD_T}\right).
\end{aligned}$$

This terminates the proof. □

With the last proof, we close the preparatory work and move on to the demonstration the result which will be the foundation to identify the first good set.

Proof of Lemma 6.9. To begin with, we dissolve the absolute value of the expression in question and obtain

$$
\begin{aligned}
&E^\star \left|\widehat{\varphi}_X^\star(u;\underline{s}) - E^\star\widehat{\varphi}_X^\star(u;\underline{s})\right|^2 \\
&= E^\star\left(\Re\left(\widehat{\varphi}_X^\star(u;\underline{s}) - E^\star\widehat{\varphi}_X^\star(u;\underline{s})\right)\right)^2 + E^\star\left(\Im\left(\widehat{\varphi}_X^\star(u;\underline{s}) - E^\star\widehat{\varphi}_X^\star(u;\underline{s})\right)\right)^2 \\
&=: \mathrm{I} + \mathrm{II}. \qquad (A.306)
\end{aligned}
$$

Since both the real and the imaginary part can be treated similarly, we limit ourselves to the examination of the cosine case, namely term I. First, we notice the centering of the expression, which allows for the transition to the bootstrap variance, that is

$$
\begin{aligned}
\mathrm{I} &= E^\star\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right. \\
&\qquad\qquad \left. - E^\star\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right)\right)^2 \\
&= \mathrm{Var}^\star\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right) \\
&= \frac{1}{b_T T}\mathrm{Var}^\star\left(\frac{1}{(b_T T)^{1/2}}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right). \qquad (A.307)
\end{aligned}
$$

The fragmentation of the prefactor in the last step is conductive to ease the calculations later on since it allows for the use of already established procedures. Therefore, we will focus on

$$
\mathrm{Var}^\star\left(\frac{1}{(b_T T)^{1/2}}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right) \qquad (A.308)
$$

from now on and incorporate the missing $(b_T T)^{-1}$ back in afterwards. Next, we make use of the block structure and the consequential independence to split the variance as seen in the proof of Theorem 3.18 such that equation (A.308) becomes

$$
\begin{aligned}
&\mathrm{Var}^\star\left(\frac{1}{(b_T T)^{1/2}}\sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right) \\
&\quad + \mathrm{Var}^\star\left(\frac{1}{(b_T T)^{1/2}}\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right) \\
&\qquad + \mathrm{Var}^\star\left(\frac{1}{(b_T T)^{1/2}}\sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}_{t,T}^\star\rangle\right)\right) \\
&=: \mathrm{Ia} + \mathrm{Ib} + \mathrm{Ic}. \qquad (A.309)
\end{aligned}
$$

Before we start the examination of the main subterm, Ib, we direct our attention to subterms Ia and Ic, which contain one endpoint group and at most one additional inchoate bootstrap block each if we assume T large enough that it holds $L_T < d_T^{-\frac{\delta}{2(1+\delta)}}$. Thus, every sum contains less than $TD_T + L_T$ summands. Due to their similarities, it suffices to concentrate on one of them. In our case, this will be Ia. By transforming the variance into a double sum of covariances, we obtain the following using Assumption 6 and Lemma 3.17:

$$\begin{aligned}
\mathrm{Ia} &= \frac{1}{b_T T} \sum_{t_1,t_2=L_T\lfloor T/L_T\rfloor+1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad\qquad \cdot \mathrm{Cov}^\star\left(\cos\left(\langle \underline{s}, \underline{X}^\star_{t_1,T}\rangle\right), \cos\left(\langle \underline{s}, \underline{X}^\star_{t_21,T}\rangle\right)\right) \\
&\leq C_{EP}\, \frac{\left(TD_T + L_T\right)^2}{b_T T} \\
&\leq C_1 \frac{\left(TD_T\right)^2}{b_T T} \\
&= \mathcal{O}\left(\frac{TD_T^2}{b_T}\right).
\end{aligned} \tag{A.310}$$

We can transfer this result directly and obtain $\mathrm{Ic} \leq \mathcal{O}\left(\frac{TD_T^2}{b_T}\right)$ also. The ensuing question is now whether these two subterms are negligible or not. To answer the question, we expand the fraction to obtain $\frac{(TD_T)^2}{b_T T}$ again. Following Assumption 6 and part (ii) of Remark 4.1, d_T^{-1} is smaller of equal to $2\left(\lfloor b_T T\rfloor + 1\right)$. Consequently, we can bound said fraction by $C\left(b_T T\right)^{-\frac{\delta}{2+\delta}}$, which tends to 0 as T tends to ∞. Thus, $\mathcal{O}\left(\frac{TD_T^2}{b_T}\right)$ is part of the class $o(1)$.

Now we move on to subterm Ib in equation (A.309). Because the sum consists of several whole bootstrap blocks, we can benefit from the independence between the single blocks. Moreover, we find ourselves in a situation comparable to the proof of Theorem 3.18. Therefore, we use the steps performed in said proof as a guideline for the proceedings in this case while adjusting the used results to our setting.

Making use of the independence of the bootstrap blocks, we rewrite subterm Ib to get

$$\begin{aligned}
\mathrm{Ib} = \frac{1}{b_T T} &\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \sum_{j=1}^{L_T}\sum_{l=1}^{L_T} K\left(\frac{\left(tL_T+j\right)/T-u}{b_T}\right) K\left(\frac{\left(tL_T+l\right)/T-u}{b_T}\right) \\
&\cdot \left(\frac{1}{2\,TD_T+1} \sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{X}_{tL_T+j+r,T}\rangle\right) \cos\left(\langle \underline{s}, \underline{X}_{tL_T+l+r,T}\rangle\right)\right. \\
&\left. -\frac{1}{\left(2\,TD_T+1\right)^2} \sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{X}_{tL_T+j+r,T}\rangle\right) \sum_{k=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{X}_{tL_T+l+k,T}\rangle\right)\right).
\end{aligned}$$

Next, we want to transform the bootstrap covariance into its real world counterpart. Thus, we start by replacing the process $\left(\underline{X}_{t,T}\right)$ with the companion one. To do so, we use Lemma A.15 and obtain

$$\begin{aligned}\mathrm{Ib} = \frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} \sum_{l=1}^{L_T} K\left(\frac{(tL_T+j)/T-u}{b_T}\right) K\left(\frac{(tL_T+l)/T-u}{b_T}\right) \\ \cdot \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\rangle\right)\right. \\ \cdot \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\rangle\right) \\ - \frac{1}{(2TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\rangle\right)\right. \\ \left.\left. \cdot \sum_{k=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\rangle\right)\right)\right) + |\underline{s}|_1 \mathcal{O}_P\left(\frac{L_T}{T}\right) \qquad \text{(A.311)}\end{aligned}$$

comparable to the proof of Theorem 3.18. Now we rewrite the three outer sums likely to the proof of Theorem 3.8. Again, we choose h to denote the lag and get for the first summand in (A.311)

$$\begin{aligned}\frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T, L_T-h\}} \\ K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \\ \cdot \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\rangle\right)\right. \\ \cdot \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\rangle\right) \\ - \frac{1}{(2TD_T+1)^2} \left(\sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\rangle\right)\right. \\ \left.\left. \cdot \sum_{k=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j+k}{T}\right)\rangle\right)\right)\right). \qquad \text{(A.312)}\end{aligned}$$

Next, we want the inner sum's dependence of the summation index to vanish. Therefore, we change the argument of $\left(\underline{\widetilde{X}}_t(u)\right)$ with the help of Lemma A.16 is used to change the argument. Thus,

$$\frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \cdot \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\rangle\right) \cdot \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\rangle\right) - \frac{1}{(2TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\rangle\right) \cdot \sum_{k=-TD_T}^{TD_T} \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\rangle\right)\right)\right) + |\underline{s}|_1\, \mathcal{O}_P\left(\frac{L_T}{T}\right) + |\underline{s}|_1\, \mathcal{O}_P\left(L_T\, D_T\right) \quad \text{(A.313)}$$

takes the place of equation (A.312). After leaving the dependence of the arguments with regard to the summation indices behind, the application of Lemma 6.8 turns the bootstrap covariance into the real world counterpart. Therefore, we have

$$\mathrm{Ib} = \frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \cdot \mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\rangle\right)\right) + |\underline{s}|_1\, \mathcal{O}_P\left(\frac{L_T}{T}\right) + |\underline{s}|_1\, \mathcal{O}_P\left(L_T\, D_T\right) + \left(|\underline{s}|_1+1\right) \mathcal{O}_P\left(\frac{L_T}{TD_T}\right) \quad \text{(A.314)}$$

to work with instead of (A.313). Like in the proof appurtenant to Lemma 2.13, a modification of the inner sum to wipe out the minimum and maximum is desirable. Hence, we show the negligibility of the difference between the first summand of (A.314) and

$$\frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=1}^{L_T} K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \cdot \mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\rangle\right), f\left(\langle \underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\rangle\right)\right) \quad \text{(A.315)}$$

as T tends to infinity. But before, we have to assure the well-definition of (A.315). In order to do so, we use parts of the notation modification seen in the proof of Lemma 2.13, namely

$$\underline{\widetilde{X}}_v(z) = \begin{cases} \underline{\widetilde{X}}_v(1), & z > 1, \\ \underline{\widetilde{X}}_v(0), & z < 0, \end{cases}$$

for all $v \in \mathbb{Z}$. Taking the absolute value of said difference leads now to

$$\begin{aligned}
&\frac{1}{b_T T} \Bigg| \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T, L_T-h\}} \\
&\qquad K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \\
&\qquad \cdot \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\rangle\right), f\left(\langle \underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\rangle\right)\right) \\
&\quad - \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=1}^{L_T} K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \\
&\qquad \cdot \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\rangle\right), f\left(\langle \underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\rangle\right)\right)\Bigg| \\
&\leq \frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=1}^{L_T-1} \sum_{j=L_T-h+1}^{L_T} K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \\
&\qquad \cdot \left|\operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\rangle\right), f\left(\langle \underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\rangle\right)\right)\right| \\
&\quad + \frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{-1} \sum_{j=1}^{-h} K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \\
&\qquad \cdot \left|\operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h\left(\frac{tL_T+j+h}{T}\right)\rangle\right), f\left(\langle \underline{s}, \underline{\widetilde{X}}_0\left(\frac{tL_T+j}{T}\right)\rangle\right)\right)\right| \\
&\leq \frac{C_2}{L_T} \sum_{h=1}^{L_T-1} h \sum_{|k| \geq \lceil h/2 \rceil} \frac{B}{l(k)} |\underline{s}|_1 \\
&= |\underline{s}|_1 \, \mathcal{O}\left(L_T^{-\tilde{\delta}}\right)
\end{aligned}$$

with the help of the proof of Theorem 3.18 and the results originating from the proof of Lemma 5.9. Because L_T tends to infinity as $T \to \infty$, $\mathcal{O}\left(L_T^{-\tilde{\delta}}\right)$ is part of $o(1)$, and the difference is negligible for fixed $\underline{s}$. Therefore, we can exchange the first summand of (A.314) with (A.315). Next, we commute the first two sums and obtain

$$\begin{aligned}\mathrm{Ib} = \frac{1}{b_T T} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{j=1}^{L_T} K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right) K\left(\frac{(tL_T+j)/T-u}{b_T}\right) \\ \cdot \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \widetilde{\underline{X}}_h\left(\frac{tL_T+j+h}{T}\right)\rangle\right), f\left(\langle \underline{s}, \widetilde{\underline{X}}_0\left(\frac{tL_T+j}{T}\right)\rangle\right)\right) \\ + |\underline{s}|_1 \mathcal{O}_P\left(\frac{L_T}{T}\right) + |\underline{s}|_1 \mathcal{O}_P\left(L_T D_T\right) + \left(|\underline{s}|_1+1\right) \mathcal{O}_P\left(\frac{L_T}{TD_T}\right) + |\underline{s}|_1 \mathcal{O}\left(L_T^{-1}\right).\end{aligned}$$

At this point, we undo one of our previous steps and incorporate the sum over j back into the sum over t. In doing so, we get

$$\begin{aligned}\mathrm{Ib} = \frac{1}{b_T T} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil}^{L_T\lfloor (T-TD_T)/L_T \rfloor -1} K\left(\frac{(t+h)/T-u}{b_T}\right) K\left(\frac{t/T-u}{b_T}\right) \\ \cdot \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \widetilde{\underline{X}}_h\left(\frac{t+h}{T}\right)\rangle\right), f\left(\langle \underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\rangle\right)\right) \\ + |\underline{s}|_1 \mathcal{O}_P\left(\frac{L_T}{T}\right) + |\underline{s}|_1 \mathcal{O}_P\left(L_T D_T\right) + \left(|\underline{s}|_1+1\right) \mathcal{O}_P\left(\frac{L_T}{TD_T}\right) + |\underline{s}|_1 \mathcal{O}\left(L_T^{-1}\right). \quad \text{(A.316)}\end{aligned}$$

Then we consult Lemma 5.9 to bound the covariance in (A.316) leading to

$$\begin{aligned}\mathrm{Ib} \leq \frac{1}{b_T T} \sum_{h=1}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil}^{L_T\lfloor (T-TD_T)/L_T \rfloor -1} K\left(\frac{(t+h)/T-u}{b_T}\right) K\left(\frac{t/T-u}{b_T}\right) \frac{C_{Cov,cs}}{h^{1+\tilde{\delta}}} |\underline{s}|_1 \\ + \frac{1}{b_T T} \sum_{h=-(L_T-1)}^{-1} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} K\left(\frac{(t+h)/T-u}{b_T}\right) K\left(\frac{t/T-u}{b_T}\right) \frac{C_{Cov,cs}}{|h|^{1+\tilde{\delta}}} |\underline{s}|_1 \\ + \frac{1}{b_T T} \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} K\left(\frac{t/T-u}{b_T}\right) K\left(\frac{t/T-u}{b_T}\right) \\ + |\underline{s}|_1 \mathcal{O}_P\left(\frac{L_T}{T}\right) + |\underline{s}|_1 \mathcal{O}_P\left(L_T D_T\right) + \left(|\underline{s}|_1+1\right) \mathcal{O}_P\left(\frac{L_T}{TD_T}\right) + |\underline{s}|_1 \mathcal{O}\left(L_T^{-1}\right). \quad \text{(A.317)}\end{aligned}$$

With the use of the uppermost number of non-zero summands due to the kernel K and the ability to be totaled appurtenant to the sum over h, we can bound the first three summands of $(A.317)$, in turn, by $C_3\left(|\underline{s}|_1+1\right)$ leading to

$$|\underline{s}|_1 \mathcal{O}_P\left(\frac{L_T}{T}\right) + |\underline{s}|_1 \mathcal{O}_P\left(L_T D_T\right) + \left(|\underline{s}|_1+1\right) \mathcal{O}_P\left(\frac{L_T}{TD_T}\right) + |\underline{s}|_1 \mathcal{O}\left(L_T^{-1}\right) + \left(|\underline{s}|_1+1\right) \mathcal{O}(1) \quad \text{(A.318)}$$

as an upper bound for equation $(A.317)$ as a whole. Thus, the next step will be to examine the rates above in view of possibilities to condense some of them. First of all, we recognize two types of terms, namely $\mathcal{O}$ and $\mathcal{O}_P$. Although boundedness implies boundedness in

P-probability, we do not incorporate the $\mathcal{O}$-terms into suitable $\mathcal{O}_P$-classes. Leaving these two types separate provides more preciseness of the bound, which will become useful later on. We begin with the subsumption of the $\mathcal{O}$-terms. Since there are two of them, we can easily detect the dominating one. Following Assumption 6, L_T^{-1} is smaller than 1. Therefore, we have

$$\mathcal{O}\left(L_T^{-1}\right) \subseteq \mathcal{O}(1).$$

As we have to take care of the prefactors, we increase the one belonging to $\mathcal{O}(1)$ by 1 and obtain

$$\left(|\underline{s}|_1 + 1\right) \mathcal{O}(1)$$

as overriding term. Then we move on to the $\mathcal{O}_P$-terms. We claim the second one of equation (A.318) to be the dominating one. In the following, we verify this assertion one by one. Before we start the comparisons of the rates, we have a look at the prefactors. Since $\left(|\underline{s}|_1 + 1\right)$ accompanies $\mathcal{O}_P\left(\frac{L_T}{TD_T}\right)$ yet, we can already secure this term as remaining prefactor. This conveys us to the rates. Because of $TD_T \leq T$, we have

$$\mathcal{O}_P\left(\frac{L_T}{T}\right) \subseteq \mathcal{O}_P\left(\frac{L_T}{TD_T}\right).$$

To verify

$$\mathcal{O}_P\left(L_T D_T\right) \subseteq \mathcal{O}_P\left(\frac{L_T}{TD_T}\right), \tag{A.319}$$

we review the following equation:

$$\frac{L_T}{TD_T} - L_T D_T = L_T\left(\frac{1}{TD_T} - D_T\right) > 0. \tag{A.320}$$

As L_T is a positive integer, we focus on the difference in brackets. An equivalent formulation is

$$\left(TD_T\right)^2 < T. \tag{A.321}$$

As stated in Assumption 6, it holds

$$\left(TD_T\right)^2 \leq \left(Td_T\right)^{\frac{2}{2+\delta}},$$

whereas $\left(Td_T\right)^{\frac{2}{2+\delta}}$ can be bounded by $T^{\frac{2}{2+\delta}}$. Since $\frac{2}{2+\delta}$ is smaller than 1 for every $\delta \in (0,1)$, (A.321) holds true and, as a consequence thereof, equation (A.320) also. With that said, (A.319) is confirmed. In conclusion, we obtain

$$\mathrm{Ib} = \left(|\underline{s}|_1 + 1\right)\left(\mathcal{O}(1) + \mathcal{O}_P\left(\frac{L_T}{TD_T}\right)\right).$$

The combination of the rates for Ia, Ib and Ic leads to

$$\mathcal{O}(1) + \left(|\underline{s}|_1 + 1\right)\left(\mathcal{O}(1) + \mathcal{O}_P\left(\frac{L_T}{TD_T}\right)\right). \tag{A.322}$$

Therefore, the next step is the simplification of this sum. First of all, since (1) is lacking a prefactor, we can endow this term with $(|\underline{s}|_1 + 1)$ without doing any harm. Secondly, we can replace $o(1)$ with $\mathcal{O}(1)$ because we already have one $\mathcal{O}(1)$-term to deal with. Thus, we incorporate this term in the second summand of equation (A.322). At this point, we rejoin the prefactor $(b_T T)^{-1}$ gives

$$\mathrm{I} = \frac{1}{b_T T} (|\underline{s}|_1 + 1) \left(\mathcal{O}(1) + \mathcal{O}_P \left(\frac{L_T}{T D_T} \right) \right).$$

As already mentioned in the beginning of the proof, we obtain the very same rate for term II of equation (A.306), and since their sum results in the same rate as well, the proof is finished. □

Because both the consistency and the result concerning the asymptotic distribution are supposed to hold in P-probability, we aim to relax the afore-proven result to the effect that we obtain an upper bound using only $\mathcal{O}$-terms in place of $\mathcal{O}_P$-ones.

The proof belonging to Lemma 6.9 shows a close resemblance to the proof of Theorem 3.18. Therefore, we adapt the way the sequence of good sets is determined in Lemma A.12 and proceed as follows:

Lemma A.17.
Let Assumption 12 be fulfilled for $k = 0$. Then, there exist subsets $\left(\breve{B}_T(\underline{s})\right)_{T\in\mathbb{N}}$ of Ω depending on $\underline{s} \in \mathbb{R}^d$ with $P\left(\breve{B}_T(\underline{s})\right) \to 1$ as $T \to \infty$, on which it holds for every $u \in [0, 1]$

$$E^\star \left| \widehat{\varphi}^\star_X(u; \underline{s}) - E^\star \widehat{\varphi}^\star_X(u; \underline{s}) \right|^2 \leq \frac{C_{GS}}{b_T T} (|\underline{s}|_1 + 1)$$

for the particular T. Here, the occurring positive constant $C_{GS} < \infty$ is not influenced either by $\underline{s}$ or by u.

Proof. Comparable to the proof of Lemma 6.9, we start by rewriting the squared absolute value to obtain

$$\begin{aligned} &E^\star \left| \widehat{\varphi}^\star_X(u; \underline{s}) - E^\star \widehat{\varphi}^\star_X(u; \underline{s}) \right|^2 \\ &= E^\star \left(\Re \left(\widehat{\varphi}^\star_X(u; \underline{s}) - E^\star \widehat{\varphi}^\star_X(u; \underline{s}) \right) \right)^2 + E^\star \left(\Im \left(\widehat{\varphi}^\star_X(u; \underline{s}) - E^\star \widehat{\varphi}^\star_X(u; \underline{s}) \right) \right)^2 \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{A.323}$$

By Assumption 6, it holds $L_T = o\left(d_T^{-\frac{\delta}{2(1+\delta)}} \right)$. Thus, there exists a finite constant $C^\star > 0$ with $L_T < C^\star d_T^{-\frac{\delta}{2(1+\delta)}}$ because L_T is naturally bounded by the number of observations. In this context, we can adopt lots of the calculations from the aforementioned proof. Regarding term I of (A.323), we get in this way

$$\mathrm{I} = E^\star\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}^\star_{t,T}\rangle\right) - E^\star\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}^\star_{t,T}\rangle\right)\right)\right)^2$$
$$= \mathrm{Var}^\star\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}^\star_{t,T}\rangle\right)\right)$$
$$= \frac{1}{b_T T}\,\mathrm{Var}^\star\left(\frac{1}{(b_T T)^{1/2}}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\cos\left(\langle \underline{s}, \underline{X}^\star_{t,T}\rangle\right)\right)$$
$$\leq \frac{1}{b_T T}\left(C_{GM,cos,a} + \breve{R}^I_T(\underline{s}) + \breve{R}^{II}_T(\underline{s}) + \breve{R}^{III}_T(\underline{s})\right)\left(|\underline{s}|_1 + 1\right). \tag{A.324}$$

Here, $\breve{R}^I_T(\underline{s}), \breve{R}^{II}_T(\underline{s})$ and $\breve{R}^{III}_T(\underline{s})$ are rest terms depending on $\underline{s}$ and fulfilling $\breve{R}^I_T(\underline{s}) = |\underline{s}|_1\,\mathcal{O}_P\left(\frac{L_T}{T}\right)$, $\breve{R}^{II}_T(\underline{s}) = |\underline{s}|_1\,\mathcal{O}_P\left(L_T\,D_T\right)$ and $\breve{R}^{III}_T(\underline{s}) = \left(|\underline{s}|_1+1\right)\mathcal{O}_P\left(\frac{L_T}{TD_T}\right)$, respectively. Note that, in contrast to the rest terms, the positive constant $C_{GM,cos,a} < \infty$ does not depend on $\underline{s}$. Besides, there is no influence of u either. This applies to the rest terms as well. In the same manner we employed the calculations made in the proof of Lemma 6.9, we avail ourselves of the method used in the proofs of Lemmata A.6 and A.12 to continue this proof with. Hence, we identify suitable subsets of Ω, on which our rest terms possess appropriate upper bounds in preference to the $\mathcal{O}_P$-ones from above. Unlikely to said proofs, we have to deal with prefactors depending on $\underline{s}$ this time. This circumstance calls for a case differentiation while defining some of the subsets. Prior to the determination of the appurtenant subset, we need to precise each rest term beginning with the first. It holds

$$\breve{R}^I_T(\underline{s}) = \Bigg|\frac{1}{b_T T}\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor - 1}\sum_{j=1}^{L_T}\sum_{l=1}^{L_T} K\left(\frac{(tL_T+j)/T-u}{b_T}\right)K\left(\frac{(tL_T+l)/T-u}{b_T}\right)$$
$$\cdot\left(\left(\frac{1}{2\,TD_T+1}\sum_{r=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{X}_{tL_T+j+r,T}\right) f\left(\underline{s}, \underline{X}_{tL_T+l+r,T}\rangle\right)\right.\right.$$
$$\left. - \frac{1}{(2\,TD_T+1)^2}\sum_{r=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{X}_{tL_T+j+r,T}\rangle\right)\sum_{k=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{X}_{tL_T+l+k,T}\rangle\right)\right)$$
$$- \left(\frac{1}{2\,TD_T+1}\sum_{r=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \widetilde{\underline{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\rangle\right)\right.$$
$$\left.\cdot\cos\left(\langle \underline{s}, \widetilde{\underline{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\rangle\right)\right)$$
$$- \frac{1}{(2\,TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \widetilde{\underline{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\rangle\right)\right)$$

$$\left.\left.\left.\left.\left.\cdot\sum_{k=-TD_T}^{TD_T}\cos\left(\left\langle\underline{s},\underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right\rangle\right)\right)\right)\right)\right|.$$

Note that the rest term vanishes if we choose $\underline{s}=\underline{0}$. Based on that, we define

$$\begin{aligned}\breve{B}_T^I(\underline{s}) &:= \begin{cases}\left\{\omega\in\Omega\,\middle|\,\breve{R}_T^I(\underline{s})\le\frac{L_T}{TD_T}|\underline{s}|_1\right\}, & \underline{s}\neq\underline{0},\\ \Omega, & \underline{s}=\underline{0},\end{cases}\\ &=\begin{cases}\left\{\omega\in\Omega\,\middle|\,\frac{TD_T}{L_T}|\underline{s}|_1^{-1}\breve{R}_T^I(\underline{s})\le 1\right\}, & \underline{s}\neq\underline{0},\\ \Omega, & \underline{s}=\underline{0}.\end{cases}\end{aligned}$$

Now we have to confirm the validity of $P\left(\breve{B}_T^I(\underline{s})\right)\to 1$ as T tends to ∞. If $\underline{s}$ equals $\underline{0}$, $P\left(\breve{B}_T^I(\underline{s})\right)$ equals $P(\Omega)$, which is 1 by definition. Thus, we concentrate on the case where $\underline{s}\neq\underline{0}$. Because of

$$P\left(\breve{B}_T^I(\underline{s})\right)\ge 1-E\left(\frac{TD_T}{L_T}|\underline{s}|_1^{-1}\breve{R}_T^I(\underline{s})\right),$$

we bound the expectation. It holds

$$E\left(\frac{TD_T}{L_T}|\underline{s}|_1^{-1}\breve{R}_T^I(\underline{s})\right)=\mathcal{O}\left(\frac{TD_T}{L_T}|\underline{s}|_1^{-1}|\underline{s}|_1\frac{L_T}{T}\right)=\mathcal{O}(D_T),$$

which gives the desired convergence $\lim_{T\to\infty}\breve{B}_T^I(\underline{s})=1$. Moving on to the second rest term of equation (A.324), we have

$$\begin{aligned}\breve{R}_T^{II}(\underline{s})=\Bigg|&\frac{1}{b_TT}\sum_{t=\lceil(TD_T+1)/L_T\rceil}^{\lfloor(T-TD_T)/L_T\rfloor-1}\sum_{j=1}^{L_T}\sum_{h=-(L_T-1)}^{L_T-1}\sum_{l=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}}\\ &K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right)K\left(\frac{(tL_T+j)/T-u}{b_T}\right)\\ &\cdot\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\cos\left(\left\langle\underline{s},\underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right\rangle\right)\right.\\ &\qquad\cdot\cos\left(\left\langle\underline{s},\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right\rangle\right)\\ &-\frac{1}{(2TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T}\cos\left(\left\langle\underline{s},\underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right\rangle\right)\right.\\ &\qquad\left.\cdot\sum_{k=-TD_T}^{TD_T}\cos\left(\left\langle\underline{s},\underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j+k}{T}\right)\right\rangle\right)\right)\\ &-\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\cos\left(\left\langle\underline{s},\underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right\rangle\right)\right.\end{aligned}$$

$$\cdot \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\rangle\right)$$
$$-\frac{1}{(2\,TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\rangle\right)\right.$$
$$\left.\left.\left.\left.\cdot\sum_{k=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\rangle\right)\right)\right)\right)\right|.$$

Similar to the previous rest term, $\breve{R}_T^{II}(\underline{s})$ disappears for $\underline{s}=\underline{0}$. Therefore, we set

$$\breve{B}_T^{II}(\underline{s}) := \begin{cases}\left\{\omega\in\Omega\,\middle|\,\breve{R}_T^{II}(\underline{s})\le L_T\,D_T^{\frac{1+\delta}{2}}\,|\underline{s}|_1\right\}, & \underline{s}\neq\underline{0},\\ \Omega, & \underline{s}=\underline{0},\end{cases}$$
$$= \begin{cases}\left\{\omega\in\Omega\,\middle|\,\left(L_T\,D_T^{\frac{1+\delta}{2}}\,|\underline{s}|_1\right)^{-1}\breve{R}_T^{II}(\underline{s})\le 1\right\}, & \underline{s}\neq\underline{0},\\ \Omega, & \underline{s}=\underline{0}.\end{cases}$$

As before, we only have to focus on the case where $\underline{s}$ is unequal to $\underline{0}$. Like above, we have

$$P\left(\breve{B}_T^{II}(\underline{s})\right)\ge 1-E\left(\left(L_T\,D_T^{\frac{1+\delta}{2}}\,|\underline{s}|_1\right)^{-1}\breve{R}_T^{IV}(\underline{s})\right).$$

Bounding the expectation via

$$E\left(\left(L_T\,D_T^{\frac{1+\delta}{2}}\,|\underline{s}|_1\right)^{-1}\breve{R}_T^{II}(\underline{s})\right)=\mathcal{O}\left(L_T^{-1}D_T^{-\frac{1+\delta}{2}}\,|\underline{s}|_1^{-1}\,|\underline{s}|_1\,L_T\,D_T\right)=\mathcal{O}\left(D_T^{\frac{1-\delta}{2}}\right)$$

leads to $\lim_{T\to\infty}\breve{B}_T^{II}(\underline{s})=1$. For the last rest term of (A.324), it holds

$$\breve{R}_T^{III}(\underline{s}) = \left|\frac{1}{b_T T}\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1}\;\sum_{h=-(L_T-1)}^{L_T-1}\;\sum_{l=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}}\right.$$
$$K\left(\frac{(tL_T+j+h)/T-u}{b_T}\right)K\left(\frac{(tL_T+j)/T-u}{b_T}\right)$$
$$\cdot\left(\frac{1}{2\,TD_T+1}\sum_{r=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\rangle\right)\right.$$
$$\cdot\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\rangle\right)$$
$$-\frac{1}{(2\,TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\rangle\right)\right.$$
$$\left.\left.\cdot\sum_{k=-TD_T}^{TD_T}\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\rangle\right)\right)\right)$$

$$- \operatorname{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j+h}\left(\frac{tL_T+j+h}{T}\right)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_{tL_T+j}\left(\frac{tL_T+j}{T}\right)\rangle\right)\right)\Bigg|.$$

This time, the rest term does not equals 0 if $\underline{s} = \underline{0}$. Consequently, a definition by cases concerning the subset is not necessary. Thus, we set

$$\begin{aligned}\breve{B}_T^{III}(\underline{s}) &:= \left\{\omega \in \Omega \,\middle|\, \breve{R}_T^{III}(\underline{s}) \leq \frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\left(|\underline{s}|_1 + 1\right)\right\} \\ &= \left\{\omega \in \Omega \,\middle|\, \frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\left(|\underline{s}|_1 + 1\right)^{-1} \breve{R}_T^{III}(\underline{s}) \leq 1\right\}.\end{aligned}$$

Again, we investigate the convergence of the belonging probability. Following

$$P\left(\breve{B}_T^{III}(\underline{s})\right) \geq 1 - E\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\left(|\underline{s}|_1 + 1\right)^{-1}\breve{R}_T^{III}(\underline{s})\right),$$

we examine the expectation and get

$$\begin{aligned}E\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\left(|\underline{s}|_1 + 1\right)^{-1}\breve{R}_T^{V}(\underline{s})\right) &= \mathcal{O}\left(\frac{(TD_T)^{\frac{4-\delta}{4}}}{L_T}\left(|\underline{s}|_1 + 1\right)^{-1}\left(|\underline{s}|_1 + 1\right)\frac{L_T}{TD_T}\right) \\ &= \mathcal{O}\left((TD_T)^{-\frac{\delta}{4}}\right).\end{aligned}$$

This leads to the sought-after result, that is $\lim_{T\to\infty} P\left(\breve{B}_T^{III}(\underline{s})\right) = 1$. At this point, we intersect the beforehand defined subsets and obtain

$$\breve{B}_{T,cos}(\underline{s}) := \breve{B}_T^{I}(\underline{s}) \cap \breve{B}_T^{II}(\underline{s}) \cap \breve{B}_T^{III}(\underline{s}) \tag{A.325}$$

with $P\left(\breve{B}_{T,cos}(\underline{s})\right) \to 1$ as $T \to \infty$. Concluding, it holds for every $\omega \in \breve{B}_{T,cos}(\underline{s})$

$$\mathrm{I} \leq \frac{1}{b_T T}\left(C_{GS,cos,a}\left(|\underline{s}|_1 + 1\right) + |\underline{s}|_1\frac{L_T}{TD_T} + |\underline{s}|_1 L_T D_T^{\frac{1+\delta}{2}} + \left(|\underline{s}|_1 + 1\right)\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}\right). \tag{A.326}$$

Now we consolidate the summands of equation (A.326). Clearly, the dominating prefactor is $(|\underline{s}|_1 + 1)$. Thus, we focus on the rates depending on T. In the following, we will show that all of these rates can be bounded by constants. Thus, the maximum of those will be the ruling one to build the overall bound with. We start with the first rate. To show $\frac{L_T}{TD_T} < C_{GS,cos,b}$ for some constant $0 < C_{GS,cos,b} < \infty$, we bound the numerator from beneath and the denominator from above. Following our assumption from the beginning, we have $L_T < C^\star d_T^{-\frac{\delta}{2(1+\delta)}}$, whereas TD_T is larger than $d_T^{-\frac{\delta}{2+\delta}}$ as written in Assumption 6. This leads to

$$C^\star d_T^{-\frac{\delta}{2(1+\delta)}} < C_{GS,cos,b}\, d_T^{-\frac{\delta}{2+\delta}}$$

as new inequality in question. Hence, we compare the exponents and verify

$$\frac{\delta}{2(1+\delta)} < \frac{\delta}{2+\delta}. \tag{A.327}$$

As this is equivalent to $\delta > 0$, $C_{GS,cos,b}$ can be chosen to dominate $\frac{L_T}{TD_T}$ for every choice of $\delta \in (0,1)$. Moving on to the second rate in question, namely $L_T\, D_T^{\frac{1+\delta}{2}}$. This term is bounded by some finite positive constant $C_{GS,cos,c}$ if L_T is either smaller than or equal to $D_T^{-\frac{1+\delta}{2}}$ disregarding potential constants, whereof the former is the case. This can be seen by using Assumption 6 to bound $D_T^{-\frac{1+\delta}{2}}$ from beneath by

$$\left(Td_T^{\frac{\delta}{2+\delta}}\right)^{\frac{1+\delta}{2}}. \tag{A.328}$$

Since it holds $T \geq d_T^{-1}$, equation (A.328), in turn, can be bounded from beneath by $d_T^{-\frac{1+\delta}{2+\delta}}$. Therefore, we have to verify

$$\frac{\delta}{2(1+\delta)} < \frac{1+\delta}{2+\delta} \tag{A.329}$$

similar to the comparison before. If equation (A.329) turns out to be true, it holds

$$d_T^{-\frac{\delta}{2(1+\delta)}} < (C^{\star})^{-1}\, d_T^{-\frac{1+\delta}{2+\delta}},$$

and we set $C_{GS,cos,c} := (C^{\star})^{-1}$. Because the right-hand side of (A.329) is larger than the right-hand side of (A.327), equation (A.329) holds true for every $\delta \in (0,1)$. Lastly, we look at $\frac{L_T}{(TD_T)^{\frac{4-\delta}{4}}}$. Bounding the denominator from beneath gives

$$(C^{\star})^{-1}\, L_T < (C^{\star})^{-1}\, d_T^{-\frac{\delta}{2+\delta}\cdot\frac{4-\delta}{4}}$$

to be verified. This leads to the following inequality

$$\frac{\delta}{2(1+\delta)} < \frac{\delta}{2+\delta}\cdot\frac{4-\delta}{4}. \tag{A.330}$$

Equation (A.330) can be simplified and becomes $0 < 2 + 2\delta - \delta^2$. Consequently, it is satisfied by any $\delta \in (0,1)$. Thus, we define $C_{GS,cos,d} := (C^{\star})^{-1}$ an an upper bound for the last summand's rate. Recapitulatory, we obtain

$$\mathrm{I} \leq \frac{C_{GS,cos}}{b_T T}\left(|\underline{s}|_1 + 1\right) \tag{A.331}$$

with

$$C_{GS,cos} := \max_{i=a,\dots,d} C_{GS,cos,i}$$

instead of (A.326). This finishes the first part. Now we go back to equation (A.323), where the second term is left to look at. If we repeat all steps made for term I, but with the sine function in place of the cosine, we obtain a suitable subset $\breve{B}_{T,sin}(\underline{s})$, whose sequence fulfills $\lim_{T\to\infty} P\left(\breve{B}_{T,sin}(\underline{s})\right) = 1$ and on which it holds

$$\text{II} \leq \frac{C_{GS,sin}}{b_T T}\left(|\underline{s}|_1 + 1\right). \tag{A.332}$$

Combining both $\breve{B}_{T,cos}(\underline{s})$ and $\breve{B}_{T,sin}(\underline{s})$ by intersection gives

$$\breve{B}_T(\underline{s}) := \breve{B}_{T,cos}(\underline{s}) \cap \breve{B}_{T,sin}(\underline{s})$$

with $P\left(\breve{B}_T(\underline{s})\right) \to 1$ as T tends to ∞. With (A.331) and (A.332) in mind, it holds for every $\omega \in \breve{B}_T(\underline{s})$

$$E^\star\left|\widehat{\varphi}_X^\star(u;\underline{s}) - E^\star\widehat{\varphi}_X^\star(u;\underline{s})\right|^2 \leq \frac{C_{GS}}{b_T T}\left(|\underline{s}|_1 + 1\right)$$

with

$$C_{GS} := \max\left\{C_{GS,cos}, C_{GS,sin}\right\}.$$

This completes the proof. □

We established a suitable good set for every choice of $\underline{s} \in \mathbb{R}^d$. However, we need a subset of Ω, which combines the afore-identified good sets. This has its reason in the integration area of (6.1). The first idea which comes in mind is to combine the good sets from Lemma A.17 via intersection. Albeit, this is not possible due to the uncountability of the real numbers. We will solve this problem by taking advantage of the continuity of the sine and cosine function in the following lemma:

Lemma A.18.
Suppose Assumption 12 is satisfied for $k = 0$. Then, there exist subsets $\left(\breve{B}_T\right)_{T\in\mathbb{N}}$ of Ω with $P\left(\breve{B}_T\right) \to 1$ as T tends to ∞, on which it holds, for the relative index T,

$$E^\star\left|\widehat{\varphi}_X^\star(u;\underline{s}) - E^\star\widehat{\varphi}_X^\star(u;\underline{s})\right|^2 \leq \frac{C_{GS}}{b_T T}\left(|\underline{s}|_1 + 1\right) \tag{A.333}$$

uniformly for $u \in [0,1]$ and all $\underline{s} \in \mathbb{R}^d$ with C_{GS} being the constant established in Lemma A.17.

Proof. According to Lemma A.17, there exist subsets $\left(\breve{B}_T(\underline{s})\right)_{T\in\mathbb{N}}$ of which each fulfills the desired T-depending inequality for the corresponding $\underline{s} \in \mathbb{R}^d$. In the following, we combine some of these subsets by intersection and show that this combination forms the subset we were looking for. Thus, consider

$$\breve{B}_T := \bigcap_{\underline{s}\in\mathbb{Q}^d} \breve{B}_T(\underline{s}). \tag{A.334}$$

Since the intersection comprises all $\underline{s} \in \mathbb{Q}^d$ in preference to $\underline{s} \in \mathbb{R}^d$, we have a countably infinite number of intersections in total. Therefore, it holds

$$P\left(\breve{B}_T\right) \longrightarrow 1$$

as T tends to ∞. Suppose now, the exist some $\omega \in \breve{B}_T$ for which the right-hand side of equation (A.333) does not portray an upper bound for the expectation on the left-hand side for some $\underline{\bar{s}} \in \mathbb{R}^d$ having at least one irrational component. In the following, we will show that this assumption leads to a contradiction. First, every irrational number can be rewritten as limit of a sequence consisting of rational numbers. If we look at vectors, this behavior can be transferred component-wise. Thus, to cabin the complexity of notation, we restrict ourselves to the one-dimensional case. By this means, we look at $\bar{s} \in \mathbb{R}\backslash\mathbb{Q}$ for which it holds

$$E^{\star}\left|\widehat{\varphi}_X^{\star}\left(u;\bar{s}\right) - E^{\star}\widehat{\varphi}_X^{\star}\left(u;\bar{s}\right)\right|^2 > \frac{C_{GS}}{b_T T}\left(\left|\bar{s}\right|_1 + 1\right) \tag{A.335}$$

for some $\omega \in \breve{B}_T$. Besides, there exists a rational sequence $\left(s_n\right)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} s_n = \bar{s}.$$

Note that both the left-hand side of (A.335) is bounded. Moreover, since they are combinations of continuous functions, they are continuous themselves. Thus, it holds

$$\lim_{n\to\infty} E^{\star}\left|\widehat{\varphi}_X^{\star}\left(u;s_n\right) - E^{\star}\widehat{\varphi}_X^{\star}\left(u;s_n\right)\right|^2 = E^{\star}\left|\widehat{\varphi}_X^{\star}\left(u;\bar{s}\right) - E^{\star}\widehat{\varphi}_X^{\star}\left(u;\bar{s}\right)\right|^2$$

and

$$\lim_{n\to\infty} \frac{C_{GS}}{b_T T}\left(\left|s_n\right|_1 + 1\right) = \frac{C_{GS}}{b_T T}\left(\left|\bar{s}\right|_1 + 1\right).$$

In addition to that, it holds

$$E^{\star}\left|\widehat{\varphi}_X^{\star}\left(u;s_n\right) - E^{\star}\widehat{\varphi}_X^{\star}\left(u;s_n\right)\right|^2 \leq \frac{C_{GS}}{b_T T}\left(\left|s_n\right|_1 + 1\right)$$

for all $\omega \in \breve{B}_T$. This leads to

$$\begin{aligned}
E^{\star}\left|\widehat{\varphi}_X^{\star}\left(u;\bar{s}\right) - E^{\star}\widehat{\varphi}_X^{\star}\left(u;\bar{s}\right)\right|^2 &= \lim_{n\to\infty} E^{\star}\left|\widehat{\varphi}_X^{\star}\left(u;s_n\right) - E^{\star}\widehat{\varphi}_X^{\star}\left(u;s_n\right)\right|^2 \\
&\leq \lim_{n\to\infty} \frac{C_{GS}}{b_T T}\left(\left|s_n\right|_1 + 1\right) \\
&= \frac{C_{GS}}{b_T T}\left(\left|\bar{s}\right|_1 + 1\right),
\end{aligned}$$

which controverts equation (A.335). In conclusion, $\left(\breve{B}_T\right)_{T\in\mathbb{N}}$ with $\breve{B}_T$ as defined in (A.334) for each T is the sought-after sequence of subsets, and the proof is finished. □

Finally, we are endowed with all results needed to prove the consistency theorem, namely Theorem 6.10.

Proof of Theorem 6.10. The procedure of this proof orientates itself to the proof of Theorem 5.13, which is the real world counterpart. As already noticed there,

$$\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) \xrightarrow{P^{\star}} 0$$

in P-probability as $T \to \infty$ is equivalent to

$$\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) \xrightarrow{d} 0$$

in P-probability as $T \to \infty$, since 0 is a constant. Comparable to said proof, we check the fulfillment of the conditions stated below in order to apply Proposition 6.3.9 of Brockwell and Davis (1991). The conditions are:

(1) For $\eta \in (0,1)$ and

$$D_\eta := \left\{ (\underline{s}_1', \underline{s}_1')' \,\middle|\, \eta \leq |\underline{s}_1|_2 \,, |\underline{s}_2|_2 \leq 1/\eta \right\},$$

it holds

$$\widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) := \int_{D_\eta} \left| \widehat{\varphi}^{\star}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - E^{\star} \widehat{\varphi}^{\star}_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) \right.$$
$$\left. + E^{\star} \widehat{\varphi}^{\star}_{Y}\left(u; \underline{s}_1\right) \, E^{\star} \widehat{\varphi}^{\star}_{Z}\left(u; \underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u; \underline{s}_1\right) \, \widehat{\varphi}^{\star}_{Z}\left(u; \underline{s}_2\right) \right|^2 d\mathfrak{w} \xrightarrow{d} 0$$

in P-probability as T tends to ∞.

(2) For $\eta \to 0$, we have

$$\mathfrak{C}^{\star}_{Y,Z;\eta}(u) \longrightarrow 0$$

P-probability.

(3) For all $\epsilon > 0$, it holds

$$\lim_{\eta \downarrow 0} \limsup_{T \to \infty} P^{\star} \left(\left| \widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) - \widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) \right| > \epsilon \right) = 0$$

in P-probability.

We handle the requisites one by one beginning with the first. The difference embodied in the integrand can be rewritten as

$$E^{\star} \widehat{\varphi}^{\star}_{Y}\left(u; \underline{s}_1\right) \, E^{\star} \widehat{\varphi}^{\star}_{Z}\left(u; \underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u; \underline{s}_1\right) \, \widehat{\varphi}^{\star}_{Z}\left(u; \underline{s}_2\right)$$
$$= E^{\star} \widehat{\varphi}^{\star}_{Y}\left(u; \underline{s}_1\right) \left(E^{\star} \widehat{\varphi}^{\star}_{Z}\left(u; \underline{s}_2\right) - \widehat{\varphi}^{\star}_{Z}\left(u; \underline{s}_2\right) \right) + \widehat{\varphi}^{\star}_{Z}\left(u; \underline{s}_2\right) \left(E^{\star} \widehat{\varphi}^{\star}_{Y}\left(u; \underline{s}_1\right) - \widehat{\varphi}^{\star}_{Y}\left(u; \underline{s}_1\right) \right), \tag{A.336}$$

and thus we have

$$\begin{aligned}\widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) \ &= \int_{D_\eta} \Big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) \\ &\qquad + E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\left(E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - \widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\right) \\ &\qquad + \widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\left(E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right) - \widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\right)\Big|^2 d\mathfrak{w}. \qquad \text{(A.337)}\end{aligned}$$

Making use of Lemmata 6.2 and 6.4 as well as Remark 6.3 in combination with Slutsky's theorem, we have the second and the third summand of the integrand in (A.337) stochastically converging to 0 as T tends to ∞ in P-probability. This convergence is uniform with respect to $(\underline{s}_1', \underline{s}_2')' \in D_\eta$. Similar to the proof of the real world counterpart, note that $\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)$ has the same building type as, for example, $\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)$. Therefore, with

$$\underline{X}^{\star}_{t,T} = \left(\underline{Y}^{\star\prime}_{t,T}, \underline{Z}^{\star\prime}_{t,T}\right)'$$

and defining

$$\underline{s} := \left(\underline{s}_1', \underline{s}_2'\right)' \qquad \text{(A.338)}$$

in combination with the continuous mapping theorem results in

$$\widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) \xrightarrow{d} 0$$

as $T \to \infty$ in P-probability. Thus, constraint (1) is settled, and we go on to (2). Likely to the real world counterpart, we rewrite $\widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u)$ and get

$$\begin{aligned}\widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) &= \int_{\mathbb{R}^p\times\mathbb{R}^q} \mathbb{1}_{D_\eta}\left(\left(\underline{s}_1', \underline{s}_2'\right)'\right) \Big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) \\ &\qquad + E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ \widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\Big|^2 d\mathfrak{w}. \qquad \text{(A.339)}\end{aligned}$$

As a consequence, the integrand in (A.339) forms a sequence in η converging to

$$\Big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) + E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ \widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\Big|^2$$

as $\eta \to 0$. Moreover, the limiting difference can again be bounded by 4, which permits us to use the dominated convergence theorem. With it, we obtain

$$\begin{aligned}&\lim_{\eta\downarrow 0} \widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) \\ &= \int_{\mathbb{R}^p\times\mathbb{R}^q} \lim_{\eta\downarrow 0} \mathbb{1}_{D_\eta}\left(\left(\underline{s}_1', \underline{s}_2'\right)'\right) \Big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) \\ &\qquad + E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ \widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\Big|^2 d\mathfrak{w} \\ &= \int_{\mathbb{R}^p\times\mathbb{R}^q} \Big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) \\ &\qquad + E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\ \widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\Big|^2 d\mathfrak{w}\end{aligned}$$

$= \widehat{\mathfrak{C}}^{\star}_{Y,Z}(u).$

Thus, the fulfillment of the second requirement is verified. Lastly, we focus on constraint (3). The application of Markov's inequality and Fubini's theorem results in

$$\begin{aligned}
&P^{\star}\Bigg(\Bigg|\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} \big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\\
&\qquad\qquad +E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)-\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\big|^2\,d\mathfrak{w}\Bigg|>\epsilon\Bigg)\\
&\leq \epsilon^{-1}E^{\star}\Bigg(\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} b_T T\,\big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\\
&\qquad\qquad +E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)-\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\big|^2\,d\mathfrak{w}\Bigg)\\
&=\epsilon^{-1}\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} E^{\star}\,\big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\\
&\qquad\qquad +E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)-\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\big|^2\,d\mathfrak{w}.
\end{aligned}\tag{A.340}$$

As we want to show the disappearance of (A.340) in P-probability, it suffices to show the existence of a subset $\breve{K}_T$ of Ω, whose sequence fulfills $P\left(\breve{K}_T\right)\to 1$ as T tends to ∞ and on which equation (A.340) vanishes. For that to happen, we consider the subsets

$$\begin{aligned}
\breve{K}^{I}_T := \Bigg\{w\in\Omega\,\Bigg|\,&E^{\star}\left|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2\\
&\leq \frac{C_{GS,I}}{b_T T}\left(\left|(\underline{s}_1',\underline{s}_2')'\right|_1+1\right)\ \forall\,(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}\Bigg\},
\end{aligned}$$

$$\breve{K}^{II}_T := \left\{w\in\Omega\,\middle|\,E^{\star}\left|\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\right|^2\leq\frac{C_{GS,II}}{b_T T}\left(|\underline{s}_2|_1+1\right)\ \forall\underline{s}_2\in\mathbb{R}^q\right\}$$

and

$$\breve{K}^{III}_T := \left\{w\in\Omega\,\middle|\,E^{\star}\left|\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)-E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\right|^2\leq\frac{C_{GS,III}}{b_T T}\left(|\underline{s}_1|_1+1\right)\ \forall\underline{s}_1\in\mathbb{R}^p\right\}.$$

Then, Lemma A.17 joint with Lemma A.18 ensures that for $T\to\infty$ it holds

$$P\left(\breve{K}^{I}_T\right)\to 1,\quad P\left(\breve{K}^{II}_T\right)\to 1\quad\text{and}\quad P\left(\breve{K}^{III}_T\right)\to 1,$$

respectively. Consequently, for a sequence formed by

$$\breve{K}_T := \breve{K}^{I}_T\cap\breve{K}^{II}_T\cap\breve{K}^{III}_T$$

it holds $P\left(\breve{K}_T\right)\to 1$ as T tends to ∞ as well. Thus, we perform the following steps on the set $\breve{K}_T$. Using (A.336), the integrand in (A.340) can be bounded by

$$E^{\star}\left(\left|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|+\left|E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)-\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\right|\right.$$
$$\left.+\left|E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)-\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\right|\right)^2. \quad \text{(A.341)}$$

Ignoring any occurring constants,

$$E^{\star}\left|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2+E^{\star}\left|E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)-\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\right|^2$$
$$+E^{\star}\left|E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)-\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\right|^2$$
$$=: \mathrm{I}+\mathrm{II}+\mathrm{III} \quad \text{(A.342)}$$

forms an upper bound for equation (A.341). Because the three newly-defined terms are built in the same way, we are able to determine upper bounds for all of them using the same method. For $\underline{s}$ defined as in (A.338), it holds

$$|\underline{s}|_1 = |\underline{s}_1|_1 + |\underline{s}_2|_1 .$$

Having that in mind, we obtain the first upper bound as follows:

$$\mathrm{I} \le \frac{C_{GS,I}}{b_T T}\left(|\underline{s}|_1+1\right) = \left(|\underline{s}_1|_1+|\underline{s}_2|_1+1\right)o(1).$$

Similarly, we

$$\mathrm{II} \le \frac{C_{GS,II}}{b_T T}\left(|\underline{s}_2|_1+1\right) = \left(|\underline{s}_2|_1+1\right)o(1)$$

and

$$\mathrm{III} \le \frac{C_{GS,III}}{b_T T}\left(|\underline{s}_1|_1+1\right) = \left(|\underline{s}_1|_1+1\right)o(1)$$

for the remaining two terms. Combining these bounds yields

$$E^{\star}\left|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)-E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right)+E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)-\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\,\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\right|^2$$
$$= o(1)\left(|\underline{s}_1|_1+|\underline{s}_2|_1+1\right)$$

for the integrand in (A.340). In consequence, we exchange the integrand with the upper bound from above and obtain

$$P^{\star}\left(\left|\widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u)-\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)\right|>\epsilon\right) = o(1)\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta}\left(|\underline{s}_1|_1+|\underline{s}_2|_1+1\right)d\mathfrak{w}.$$

According to Assumption 10, the right-hand side tends to 0 as T tends to ∞. In consequence of the underlying $\breve{K}_T$, the convergence holds in P-probability. As signalized before, the application of Proposition 6.3.9 of Brockwell and Davis (1991) is now possible, which brings this proof to a conclusion. □

The last proof marks the end of the first subsection, and we move on to the one dealing with the asymptotic distribution of $\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$.

A.5.2. Proofs of Section 6.3

This subsection occupies itself with the proofs of the findings in Section 6.3.

Similar to the real world counterpart, we are in need of some further convergence findings to prove the result attending the asymptotic distribution of $\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$. The following lemma is needed later on to cope with convergence of products consisting of bootstrap ECFs.

Lemma A.19.
Under Assumption 12 for $k = 1$ for $\left(\underline{Y}_{t,T}\right)_{t=1}^{T}$ and $\left(\underline{Z}_{t,T}\right)_{t=1}^{T}$, it holds for any $\underline{s}_1 \in [-S_1, S_1]^p$ for $S_1 \in \mathbb{R}_+$, $\underline{s}_2 \in [-S_2, S_2]^q$ with $S_2 \in \mathbb{R}_+$ and $u \in [0,1]$

$$\left((b_T T)^{1/2} \begin{pmatrix} \Re\left(\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right) - E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\right) \\ \Im\left(\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right) - E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\right) \\ \Re\left(\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\right) \\ \Im\left(\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\right) \end{pmatrix} \right)_{(\underline{s}_1,\underline{s}_2)\in[-S_1,S_1]^p\times[-S_2,S_2]^q} \xrightarrow{d} \begin{pmatrix} G_{Y,\Re}\left(u;\underline{s}_1\right) \\ G_{Y,\Im}\left(u;\underline{s}_1\right) \\ G_{Z,\Re}\left(u;\underline{s}_2\right) \\ G_{Z,\Im}\left(u;\underline{s}_2\right) \end{pmatrix}_{(\underline{s}_1,\underline{s}_2)\in[-S_1,S_1]^p\times[-S_2,S_2]^q}$$

in P-probability as $T \to \infty$. Here, $G_{Y,\Re}\left(u;\underline{s}_1\right), G_{Y,\Im}\left(u;\underline{s}_1\right), G_{Z,\Re}\left(u;\underline{s}_2\right)$ and $G_{Z,\Im}\left(u;\underline{s}_2\right)$ are centered Gaussian processes with covariance functions $\sigma^2_{Y,\Re}\left(u;\underline{s}_1,\underline{s}_1^{\circ}\right), \sigma^2_{Y,\Im}\left(u;\underline{s}_1,\underline{s}_1^{\circ}\right)$, $\sigma^2_{Z,\Re}\left(u;\underline{s}_2,\underline{s}_2^{\circ}\right)$ and $\sigma^2_{Z,\Im}\left(u;\underline{s}_2,\underline{s}_2^{\circ}\right)$, respectively, having their roots in Lemma 5.2.

Proof. The proof can be carried out analogously to the one of Theorem 3.23 under the use of the Cramér-Wold theorem for the belonging CLT. □

Eventually, we focus on the the asymptotic distribution of $\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$ while proving the main result of Section 6.3 in the following lines.

Proof of Theorem 6.11. Similar to the proof of this theorem's real world counterpart, that is Theorem 5.14, we proceed in three steps verifying the conditions needed by Proposition 6.3.9 in Brockwell and Davis (1991) to prove

$$\begin{aligned} & b_T T\, \widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) \\ & = b_T T \int_{\mathbb{R}^p\times\mathbb{R}^q} \Big|\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}\left(u;\underline{s}_1,\underline{s}_2\right) \\ & \qquad + E^{\star}\widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right) E^{\star}\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right) - \widehat{\varphi}^{\star}_{Y}\left(u;\underline{s}_1\right)\widehat{\varphi}^{\star}_{Z}\left(u;\underline{s}_2\right)\Big|^2 d\mathfrak{w} \\ & \xrightarrow{d} \int_{\mathbb{R}^p\times\mathbb{R}^q} \left|G\left(u;\underline{s}_1,\underline{s}_2\right)\right|^2 d\mathfrak{w}. \end{aligned}$$

The conditions are similar to those in the proof of Theorem 5.14, but this time, we need their validity in P-probability. The conditions read as follows:

(1) For $\eta \in (0,1)$ and

$$D_\eta := \left\{ (\underline{s}_1', \underline{s}_1')' \,\middle|\, \eta \leq |\underline{s}_1|_2 \,, |\underline{s}_2|_2 \leq 1/\eta \right\},$$

it holds

$$\begin{aligned}
&b_T T\, \widehat{\mathfrak{C}}^\star_{Y,Z;\eta}(u) \\
&:= b_T T \int_{D_\eta} \left| \widehat{\varphi}^\star_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - E^\star \widehat{\varphi}^\star_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) \right. \\
&\qquad\qquad \left. + E^\star \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\, E^\star \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\, \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) \right|^2 d\mathfrak{w} \\
&\xrightarrow{d} \int_{D_\eta} \left| G\left(u; \underline{s}_1, \underline{s}_2\right) \right|^2 d\mathfrak{w}
\end{aligned}$$

as T tends to ∞.

(2) For $\eta \to 0$, we have

$$\mathcal{G}_\eta := \int_{D_\eta} \left| G\left(u; \underline{s}_1, \underline{s}_2\right) \right|^2 d\mathfrak{w} \xrightarrow{d} \int_{\mathbb{R}^p \times \mathbb{R}^q} \left| G\left(u; \underline{s}_1, \underline{s}_2\right) \right|^2 d\mathfrak{w} =: \mathcal{G}.$$

(3) For all $\epsilon > 0$, it holds

$$\begin{aligned}
\lim_{\eta \downarrow 0} \limsup_{T \to \infty} P^\star \Bigg(\Bigg| b_T T \int_{\mathbb{R}^p \times \mathbb{R}^q \setminus D_\eta} &\left| \widehat{\varphi}^\star_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - E^\star \widehat{\varphi}^\star_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) \right. \\
&\left. + E^\star \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\, E^\star \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\, \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) \right|^2 d\mathfrak{w} \Bigg| > \epsilon \Bigg) = 0.
\end{aligned}$$

If all of these requirements hold in P-probability in each case, they hold together in P-probability and thus the looked-for convergence also. Starting with (1), we rewrite the second half of the integrand in order to create suitable differences, that is

$$\begin{aligned}
&E^\star \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\, E^\star \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\, \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) \\
&\quad = E^\star \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right) \left(E^\star \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right)\right) + \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) \left(E^\star \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right) - \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\right).
\end{aligned} \tag{A.343}$$

Thus, we can rewrite $\widehat{\mathfrak{C}}^\star_{Y,Z;\eta}(u)$ as well and obtain

$$\begin{aligned}
&b_T T\, \widehat{\mathfrak{C}}^\star_{Y,Z;\eta}(u) \\
&= \int_{D_\eta} \Big| \left(b_T T\right)^{1/2} \left(\widehat{\varphi}^\star_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) - E^\star \widehat{\varphi}^\star_{Y,Z}\left(u; \underline{s}_1, \underline{s}_2\right) \right) \\
&\qquad + \left(b_T T\right)^{1/2} E^\star \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right) \left(E^\star \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) - \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right)\right) \\
&\qquad\qquad + \left(b_T T\right)^{1/2} \widehat{\varphi}^\star_Z\left(u; \underline{s}_2\right) \left(E^\star \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right) - \widehat{\varphi}^\star_Y\left(u; \underline{s}_1\right)\right) \Big|^2 d\mathfrak{w}.
\end{aligned}$$

Similar to the proof of Theorem 5.14, we continue by the partially partitioning of the above-written integrand with regard to real and imaginary parts of the bootstrap ECF. Thereby, we have

$$\begin{aligned}
b_T T\, \widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) = \int_{D_\eta} \Big| & (b_T T)^{1/2}\, \Re\left(\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2)\right) \\
& + i\,(b_T T)^{1/2}\, \Im\left(\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2)\right) \\
& + (b_T T)^{1/2}\, E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\, \Re\left(E^{\star}\widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2) - \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\right) \\
& + i\,(b_T T)^{1/2}\, E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\, \Im\left(E^{\star}\widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2) - \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\right) \\
& + (b_T T)^{1/2}\, \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\, \Re\left(E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1) - \widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\right) \\
& + i\,(b_T T)^{1/2}\, \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\, \Im\left(E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1) - \widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\right)\Big|^2 d\mathfrak{w}. \qquad (A.344)
\end{aligned}$$

Next, we examine the summands forming the integrand in equation (A.344). Since D_η forms a compact space, Lemma A.19 in combination with part (ii) of Lemma 6.2 is applicable. With Slutsky's theorem, this leads to

$$(b_T T)^{1/2}\, E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\, \Re\left(E^{\star}\widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2) - \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\right) \xrightarrow{d} \varphi_Y(u;\underline{s}_1)\, G_{Z,\Re}(u;\underline{s}_2)$$

and

$$(b_T T)^{1/2}\, E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\, \Im\left(E^{\star}\widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2) - \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\right) \xrightarrow{d} \varphi_Y(u;\underline{s}_1)\, G_{Z,\Im}(u;\underline{s}_2)$$

in P-probability for $T \to \infty$. Moving on to the last two summands in (A.344), we use Lemma A.19 together with the first part of Lemma 6.2 and Slutsky's theorem to obtain

$$(b_T T)^{1/2}\, \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\, \Re\left(E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1) - \widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\right) \xrightarrow{d} \varphi_Z(u;\underline{s}_2)\, G_{Y,\Re}(u;\underline{s}_1)$$

and

$$(b_T T)^{1/2}\, \widehat{\varphi}^{\star}_{Z}(u;\underline{s}_2)\, \Im\left(E^{\star}\widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1) - \widehat{\varphi}^{\star}_{Y}(u;\underline{s}_1)\right) \xrightarrow{d} \varphi_Y(u;\underline{s}_1)\, G_{Z,\Im}(u;\underline{s}_2)$$

in P-probability as T tends to ∞. As before, these convergences are uniform regarding $(\underline{s}_1,\underline{s}_2)' \in D_\eta$. To treat the remaining two summands, we use the same technique as in the proofs of Theorem 5.13 and Theorem 5.14 with

$$\underline{X}^{\star}_{t,T} = \left(\underline{Y}^{\star\prime}_{t,T}, \underline{Z}^{\star\prime}_{t,T}\right)' \quad \text{and} \quad \underline{s} := (\underline{s}_1', \underline{s}_2')'. \qquad (A.345)$$

With that, we reformulate the differences in question and get

$$(b_T T)^{1/2}\, \Re\left(\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2)\right) = (b_T T)^{1/2}\, \Re\left(\widehat{\varphi}^{\star}_{X}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{X}(u;\underline{s})\right)$$

as well as

$$(b_T T)^{1/2}\, \Im\left(\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2)\right) = (b_T T)^{1/2}\, \Im\left(\widehat{\varphi}^{\star}_{X}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{X}(u;\underline{s})\right).$$

Then, a vectorized version of Theorem 3.23, similar to the one in Lemma A.19, which can be shown using the Cramér-Wold theorem in the corresponding CLT comparably to Theorem 5.6, says

$$(b_T T)^{1/2}\, \Re \left(\widehat{\varphi}^{\star}_X(u;\underline{s}) - \varphi^{\star}_X(u;\underline{s})\right) \xrightarrow{d} G_{X,\Re}(u;\underline{s})$$

and

$$(b_T T)^{1/2}\, \Im \left(\widehat{\varphi}^{\star}_X(u;\underline{s}) - \varphi^{\star}_X(u;\underline{s})\right) \xrightarrow{d} G_{X,\Im}(u;\underline{s})$$

hold in P-probability as $T \to \infty$ uniformly in $\underline{s} \in D_\eta$. Combining the recently established convergences with the continuous mapping theorem directs to the convergence we were aiming for, to wit

$$b_T T\, \widehat{\mathfrak{C}}^{\star}_{Y,Z;\eta}(u) \xrightarrow{d} \int_{D_\eta} \left|G(u;\underline{s}_1,\underline{s}_2)\right|^2 d\mathfrak{w}$$

in P-probability as T tends to ∞. The centered Gaussian process $\left(G(u;\underline{s}_1,\underline{s}_2)\right)_{(\underline{s}_1,\underline{s}_2)\in\mathbb{R}^p\times\mathbb{R}^q}$ is, as in the proof of Theorem 5.14, composed of the Gaussian processes originating from the convergences of the summands of the integrand of (A.344). More precisely, we have again

$$\begin{aligned}
&G(u;\underline{s}_2,\underline{s}_1,)\\
&= G_{X,\Re}(u;\underline{s}) + iG_{X,\Im}(u;\underline{s}) + \Re\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Re}(u;\underline{s}_2) + i\,\Im\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Re}(u;\underline{s}_2)\\
&\qquad + i\,\Re\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Im}(u;\underline{s}_2) - \Im\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Im}(u;\underline{s}_2)\\
&\qquad + \Re\,\varphi_Z(u;\underline{s}_2)\, G_{Y,\Re}(u;\underline{s}_1) + i\,\Im\,\varphi_Z(u;\underline{s}_2)\, G_{Z,\Re}(u;\underline{s}_1)\\
&\qquad\qquad + i\,\Re\,\varphi_Z(u;\underline{s}_2)\, G_{Y,\Im}(u;\underline{s}_1) - \Im\,\varphi_Z(u;\underline{s}_2)\, G_{Z,\Im}(u;\underline{s}_1)\\
&= G_{X,\Re}(u;\underline{s}) + \Re\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Re}(u;\underline{s}_2) - \Im\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Im}(u;\underline{s}_2)\\
&\qquad + \Re\,\varphi_Z(u;\underline{s}_2)\, G_{Y,\Re}(u;\underline{s}_1) - \Im\,\varphi_Z(u;\underline{s}_2)\, G_{Z,\Im}(u;\underline{s}_1)\\
&\quad + i\left(G_{X,\Im}(u;\underline{s}) + \Im\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Re}(u;\underline{s}_2) + \Re\,\varphi_Y(u;\underline{s}_1)\, G_{Z,\Im}(u;\underline{s}_2)\right.\\
&\qquad\qquad \left. + \Im\,\varphi_Z(u;\underline{s}_2)\, G_{Z,\Re}(u;\underline{s}_1) + \Re\,\varphi_Z(u;\underline{s}_2)\, G_{Y,\Im}(u;\underline{s}_1)\right)\\
&=: G_{\Re}(u;\underline{s}_1,\underline{s}_2) + i\, G_{\Im}(u;\underline{s}_1,\underline{s}_2)\,.
\end{aligned} \tag{A.346}$$

We continue with requirement (2). Here, we can completely refer to the corresponding part of the proof of Theorem 5.14 because we stand already in the real world. Thus, we move directly on to the remaining requirement (3). We make use of Markov's inequality and, anew, Fubini's theorem, to get

$$\begin{aligned}
&P^{\star}\left(\left| b_T T \int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} \left|\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2)\right.\right.\right.\\
&\qquad\qquad \left.\left.\left. + E^{\star}\widehat{\varphi}^{\star}_Y(u;\underline{s}_1)\; E^{\star}\widehat{\varphi}^{\star}_Z(u;\underline{s}_2) - \widehat{\varphi}^{\star}_Y(u;\underline{s}_1)\; \widehat{\varphi}^{\star}_Z(u;\underline{s}_2)\right|^2 d\mathfrak{w}\right| > \epsilon\right)\\
&= P^{\star}\left(\int_{\mathbb{R}^p\times\mathbb{R}^q\setminus D_\eta} b_T T \left|\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u;\underline{s}_1,\underline{s}_2)\right.\right.\\
&\qquad\qquad \left.\left. + E^{\star}\widehat{\varphi}^{\star}_Y(u;\underline{s}_1)\; E^{\star}\widehat{\varphi}^{\star}_Z(u;\underline{s}_2) - \widehat{\varphi}^{\star}_Y(u;\underline{s}_1)\; \widehat{\varphi}^{\star}_Z(u;\underline{s}_2)\right|^2 d\mathfrak{w} > \epsilon\right)
\end{aligned}$$

$$\leq \epsilon^{-1} E^{\star} \left(\int_{\mathbb{R}^p \times \mathbb{R}^q \setminus D_\eta} b_T T \left| \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) - E^{\star} \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) \right. \right.$$
$$\left. \left. + E^{\star} \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) \; E^{\star} \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) - \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) \; \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) \right|^2 d\mathfrak{w} \right)$$
$$= \epsilon^{-1} \int_{\mathbb{R}^p \times \mathbb{R}^q \setminus D_\eta} b_T T E^{\star} \left| \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) - E^{\star} \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) \right.$$
$$\left. + E^{\star} \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) \; E^{\star} \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) - \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) \; \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) \right|^2 d\mathfrak{w}. \tag{A.347}$$

Following the argumentation in the proof of Theorem 6.10, we consider the subsets

$$\breve{K}^{I}_T := \left\{ w \in \Omega \,\middle|\, E^{\star} \left| \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) - E^{\star} \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) \right|^2 \right.$$
$$\left. \leq \frac{C_{GS,I}}{b_T T} \left(\left| (\underline{s}_1', \underline{s}_2')' \right|_1 + 1 \right) \; \forall (\underline{s}_1', \underline{s}_2')' \in \mathbb{R}^{p+q} \right\},$$

$$\breve{K}^{II}_T := \left\{ w \in \Omega \,\middle|\, E^{\star} \left| \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) - E^{\star} \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) \right|^2 \leq \frac{C_{GS,II}}{b_T T} \left(|\underline{s}_2|_1 + 1 \right) \; \forall \underline{s}_2 \in \mathbb{R}^q \right\}$$

and

$$\breve{K}^{III}_T := \left\{ w \in \Omega \,\middle|\, E^{\star} \left| \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) - E^{\star} \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) \right|^2 \leq \frac{C_{GS,III}}{b_T T} \left(|\underline{s}_1|_1 + 1 \right) \; \forall \underline{s}_1 \in \mathbb{R}^p \right\}$$

once again including their combination by intersection $\breve{K}_T$ to show the vanishing of (A.347) in P-probability. As already seen in said proof, it holds $P\left(\breve{K}_T\right) \to 1$ as $T \to \infty$ due to Lemmata A.17 and A.18. Anew, we perform the following steps with $\breve{K}_T$ as a basis. Now we can bound the integrand from above using equation (A.343), which yields

$$b_T T E^{\star} \left(\left| \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) - E^{\star} \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) \right| \left| E^{\star} \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) - \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) \right| \right.$$
$$\left. + \left| E^{\star} \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) - \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) \right| \right)^2 \tag{A.348}$$

to proceed with. The expression in (A.348), in turn, can be bounded by

$$b_T T \left(E^{\star} \left| \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) - E^{\star} \widehat{\varphi}^{\star}_{Y,Z}(u; \underline{s}_1, \underline{s}_2) \right|^2 + b_T T E^{\star} \left| E^{\star} \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) - \widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) \right|^2 \right.$$
$$\left. + \; b_T T E^{\star} \left| E^{\star} \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) - \widehat{\varphi}^{\star}_{Y}(u; \underline{s}_1) \right|^2 \right)$$
$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{A.349}$$

regardless of any occurring constants. Using the bounds settled in the definition of $\breve{K}_T$, we obtain

$$\mathrm{I} \leq b_T T \frac{C_{GS,I}}{b_T T} \left(|\underline{s}|_1 + 1 \right) = \left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1 \right) \mathcal{O}(1)$$

as well as

$$\mathrm{II} \le b_T T \frac{C_{GS,II}}{b_T T} \left(|\underline{s}_2|_1 + 1\right) = \left(|\underline{s}_2|_1 + 1\right) \mathcal{O}(1)$$

and

$$\mathrm{III} \le b_T T \frac{C_{GS,III}}{b_T T} \left(|\underline{s}_1|_1 + 1\right) \left(|\underline{s}_1|_1 + 1\right) \mathcal{O}(1)$$

for the summands in equation (A.349). This results in

$$\left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1\right) \mathcal{O}(1)$$

as an upper bound for the integrand in equation (A.347). Replacing said integrand with the upper bound leads to

$$\mathcal{O}(1) \int_{\mathbb{R}^p \times \mathbb{R}^q \setminus D_\eta} \left(|\underline{s}_1|_1 + |\underline{s}_2|_1 + 1\right) d\mathfrak{w}.$$

By Assumption 10, the right-hand side tends to 0 as η tends to 0. Again, because of the consideration based on $\breve{K}_T$, the convergence holds in P-probability. Thus, the subsequent use of Proposition 6.3.9 of Brockwell and Davis (1991) terminates the proof. □

The last demonstration brings this subsection to completion and, at the same time, the whole chapter dedicated to proofs.

B | Codes and Expediting Calculations

This chapter contains the codes used to conduct the simulations in Chapters 4 and 7. Moreover, to shorten the runtime for each simulation, additional calculations for the weighted CF-distance and its empirical and bootstrap counterparts are included.

B.1. Codes Belonging to Chapter 4

In this section, we provide the codes for the box plots in Section 4.2 and the tables in Section 4.3.

B.1.1. Code for the Box Plots

The code for the box plots in Section 4.2 using $\delta_1 = 0.12, \delta_2 = 0.79, \delta_3 = 0.35$ and $T = 1000000$ as well as $N_{BS} = 50000$ plus $s = 0.6$ reads as follows:

```
set.seed(1)

r<-function(x){0.9*sin(2*pi*x)}

K<-function(x){if(x>1 | x<(-1))
  0
  else
    (3/4)*(1-x^2)
}

library(stabledist)
library(purrr)

delta1<-0.12
delta2<-0.79
delta3<-0.35
alpha<-1.5
gamma_roh<-0.5
mu<-0
beta<-0

gamma<-(gamma_roh)^(1/alpha)
gamma_schlange<-function(x){gamma_roh/(1-abs(r(x))^alpha)}

T<-100000
s<-0.6
b<-T^(-delta3)
u<-c(0.2,0.4,0.6,0.8)
```

```

phi_schlange<-function(x,y){exp(-gamma_schlange(u[x])*y^alpha)}
Karg<-function(x,y){(x/T-u[y])/b}

F_K<-2*floor(T^(1-delta3))+1

Kc<-c(ceiling(T*(u[1]-b)),ceiling(T*(u[2]-b)),ceiling(T*(u[3]-b)),ceiling(T*(u[4]-b)))
Kf<-c(floor(T*(u[1]+b)),floor(T*(u[2]+b)),floor(T*(u[3]+b)),floor(T*(u[4]+b)))
Kc1<-Kc-1
tvec<-mat.or.vec(F_K,4)

for(j in 1:4)
{
  tvec[,j]<-seq(Kc[j],Kf[j],1)
}

Argvec<-mat.or.vec(F_K,4)
Kvec<-mat.or.vec(F_K,4)

for(j in 1:4)
{
  Argvec[,j]<-sapply(tvec[,j],function(x){Karg(x,y=j)})
  Kvec[,j]<-sapply(Argvec[,j],FUN=K)
}

phi_hat<-function(Y,j){(Kvec[,j]%*%Y[tvec[,j]])/(b*T)}
phi_hat_stern<-function(Y,j){(Kvec[,j]%*%Y_stern[tvec[,j]])/(b*T)}

phi_schlange_vec<-c(phi_schlange(1,s),phi_schlange(2,s),phi_schlange(3,s),
                    phi_schlange(4,s))

L<-floor(T^delta1)
F_in<-floor(T^(1-delta2))
F<-2*F_in+1

BS_u<-c(ceiling((Kc[1]-F_in)/L),ceiling((Kc[2]-F_in)/L),ceiling((Kc[3]-F_in)/L),
        ceiling((Kc[4]-F_in)/L))
BS_o<-c(ceiling((Kf[1]+F_in)/L),ceiling((Kf[2]+F_in)/L),ceiling((Kf[3]+F_in)/L),
        ceiling((Kf[4]+F_in)/L))
BS_min<-min(BS_u)
BS_max<-max(BS_o)
BS_d<-BS_o-BS_u
BS_g<-max(BS_d)+1

Estern_phi_hat<-rep(0,4)

N_BS<-50000

X<-double(T)
Y<-double(T)

eps<-rstable(T,alpha,beta,gamma,mu)

X[1]<-r(1/T)*rstable(1,alpha,beta,gamma,mu)+eps[1]

for(i in 2:T)
{
  X[i]<-r(i/T)*X[i-1]+eps[i]
}

Y<-exp(1i*s*X)

phi_hat_vec<-c(phi_hat(Y,1),phi_hat(Y,2),phi_hat(Y,3),phi_hat(Y,4))

diff<-(b*T)^(1/2)*(phi_hat_vec-phi_schlange_vec)

for(j in 1:4)
{
  for(l in -F_in:F_in)
  {
```

```
    for(i in 1:F_K)
    {
      Estern_phi_hat[j]<-Estern_phi_hat[j]+Kvec[i,j]*exp(1i*s*X[Kc1[j]+i+1])
    }
  }
}

Estern_phi_hat<-Estern_phi_hat/(b*T*F)

phi_hat_stern_vec<-mat.or.vec(N_BS,4)
diff_stern<-mat.or.vec(N_BS,4)

for(n in 1:N_BS)
{
  Y_stern<-double(T)
  k<-rdunif(BS_g,F_in,-F_in)

  for(j in 1:4)
  {
    for(i in BS_u[j]:BS_o[j])
    {
      Y_stern[(1+(i-1)*L):(i*L)]<-Y[(1+(i-1)*L+k[i-BS_u[j]+1]):(i*L+k[i-BS_u[j]+1])]
    }
  }

  phi_hat_stern_vec[n,]<-c(phi_hat_stern(Y_stern,1),phi_hat_stern(Y_stern,2),
                           phi_hat_stern(Y_stern,3),phi_hat_stern(Y_stern,4))

  diff_stern[n,]<-(b*T)^(1/2)*(phi_hat_stern_vec[n,]-Estern_phi_hat)
}

par(mfrow=c(2,2))
boxplot(Re(diff_stern[,1]), xlab="Real part, u=0.2")
abline(h=Re(diff[1]), col='red')
boxplot(Re(diff_stern[,2]), xlab="Real part, u=0.4")
abline(h=Re(diff[2]), col='red')
boxplot(Re(diff_stern[,3]), xlab="Real part, u=0.6")
abline(h=Re(diff[3]), col='red')
boxplot(Re(diff_stern[,4]), xlab="Real part, u=0.8")
abline(h=Re(diff[4]), col='red')

par(mfrow=c(2,2))
boxplot(Im(diff_stern[,1]), xlab="Imaginary part, u=0.2")
abline(h=Im(diff[1]), col='red')
boxplot(Im(diff_stern[,2]), xlab="Imaginary part, u=0.4")
abline(h=Im(diff[2]), col='red')
boxplot(Im(diff_stern[,3]), xlab="Imaginary part, u=0.6")
abline(h=Im(diff[3]), col='red')
boxplot(Im(diff_stern[,4]), xlab="Imaginary part, u=0.8")
abline(h=Im(diff[4]), col='red')
```

Now we move on the sources of the tables in the subsequent section.

B.1.2. Codes for the results listed in the tables

Instead of presenting several codes belonging to the tables presented in Section 4.3, which differ only in the parameter choice, we show one of them in an exemplary way below. At it, we opt for the code using $\delta_1 = 0.12, \delta_2 = 0.79$ and $\delta_3 = 0.4$ as well as $T = 1000000$ and $u = 0.4$, while N and N_{BS} take the values 1000 and 2000, respectively. As before, s is fixed to 0.6.

```
set.seed(1)

r<-function(x){0.9*sin(2*pi*x)}

K<-function(x){ if(x>1 | x<(-1))
   0
   else
      (3/4)*(1-x^2)
}

library(stabledist)
library(purrr)

delta1<-0.12
delta2<-0.79
delta3<-0.4
alpha<-1.5
gamma_roh<-0.5
mu<-0
beta<-0

gamma<-(gamma_roh)^(1/alpha)

T<-1000000
s<-0.6
b<-T^(-delta3)
u<-0.4

Karg<-function(x){(x/T-u)/b}
Kc<-ceiling(T*(u-b))
Kf<-floor(T*(u+b))
Kc1<-Kc-1
tvec<-seq(Kc,Kf,1)
len<-length(tvec)
Argvec<-sapply(tvec,FUN=Karg)
Kvec<-sapply(Argvec,FUN=K)

gamma_schlange<-gamma_roh/(1-abs(r(u))^alpha)
phi_schlange<-exp(-gamma_schlange*s^alpha)

L<-floor(T^delta1)
F_in<-floor(T^(1-delta2))
F<-2*F_in+1

BS_u<-ceiling((Kc-F_in)/L)
BS_o<-ceiling((Kf+F_in)/L)
BS_g<-BS_o-BS_u+1

N<-1000
N_BS<-2000

phi_hat<-double(N)
diff<-double(N)
Estern_phi_hat<-rep(0,N)
quRe<-double(N)
qoRe<-double(N)
quIm<-double(N)
qoIm<-double(N)
quAbs<-double(N)
qoAbs<-double(N)
Cov<-0
CovRe<-0
CovIm<-0
CovAbs<-0

for(m in 1:N)
{
   X<-double(T)
   Y<-double(T)

```

```
 eps<-rstable(T,alpha,beta,gamma,mu)

X[1]<-r(1/T)*rstable(1,alpha,beta,gamma,mu)+eps[1]

 for(i in 2:T)
 {
   X[i]<-r(i/T)*X[i-1]+eps[i]
 }

 Y<-exp(1i*s*X)

 phi_hat[m]<-(Kvec%*%Y[tvec])/(b*T)

 diff[m]<-(b*T)^(1/2)*(phi_hat[m]-phi_schlange)

 for(l in -F_in:F_in)
 {
   for(i in 1:len)
   {
     Estern_phi_hat[m]<-Estern_phi_hat[m]+Kvec[i]*exp(1i*s*X[Kc1+i+l])
   }
 }

 Estern_phi_hat[m]<-Estern_phi_hat[m]/(b*T*F)

 phi_hat_stern<-double(N_BS)
 diff_stern<-double(N_BS)

 for(n in 1:N_BS)
 {
   Y_stern<-double(T)
   k<-rdunif(BS_g,F_in,-F_in)

   for(i in BS_u:BS_o)
   {
     Y_stern[(1+(i-1)*L):(i*L)]<-Y[(1+(i-1)*L+k[i-BS_u+1]):(i*L+k[i-BS_u+1])]
   }

   phi_hat_stern[n]<-(Kvec%*%Y_stern[tvec])/(b*T)
   diff_stern[n]<-(b*T)^(1/2)*(phi_hat_stern[n]-Estern_phi_hat[m])
 }

 C<-0

 quRe[m]<-quantile(Re(diff_stern),0.025)
 qoRe[m]<-quantile(Re(diff_stern),0.975)

 quIm[m]<-quantile(Im(diff_stern),0.025)
 qoIm[m]<-quantile(Im(diff_stern),0.975)

 quAbs[m]<-quantile(abs(diff_stern),0.025)
 qoAbs[m]<-quantile(abs(diff_stern),0.975)

 if(Re(diff[m])>=quRe[m] & Re(diff[m])<=qoRe[m])
 {
   CovRe<-CovRe+1
   C<-C+1
 }

 if(Im(diff[m])>=quIm[m] & Im(diff[m])<=qoIm[m])
 {
   CovIm<-CovIm+1
   C<-C+1
 }

 if(abs(diff[m])>=quAbs[m] & abs(diff[m])<=qoAbs[m])
 {
   CovAbs<-CovAbs+1
 }

```

```
  if(C==2)
  {
      Cov<-Cov+1
  }
}

CovProzRe<-CovRe/N
CovProzIm<-CovIm/N
CovProzAbs<-CovAbs/N
CovProz<-Cov/N

CovProzRe
CovProzIm
CovProzAbs
CovProz
```

This closes the section providing codes for Chapter 4. In the next, we aim at Chapter 7.

B.2. Codes and Calculations belonging to Chapter 7

In this section, we turn our attention to the simulation study conducted in Chapter 7. Before we focus on the different codes in the second subsection, some supplementary calculations are presented in the first.

B.2.1. Expediting Calculations

This subsection addresses the additional calculations used to rewrite the different versions of the weighted CF-distance.

In order to shorten the simulation time, we want to avoid the use of the *integrate*-function implemented in R, which is the program we employed to perform our simulations with. Because we run our simulations based on dimension $p = q = 1$, we are able to state the integral explicitly. However, this requires some calculations beforehand. Since our parameter choice allows us to leave the endpoints uncared, we focus on the non-endpoint case in the upcoming calculations as well. The term we start with is $b_T T\,\widehat{\mathfrak{C}}_{Y,Z}(u)$ or, to be precise,

$$\begin{aligned} b_T T\,\widehat{\mathfrak{C}}_{Y,Z}(u) &= b_T T \int_{\mathbb{R}\times\mathbb{R}} \left|\widehat{\varphi}_{Y,Z}\left(u; s_1, s_2\right) - \widehat{\varphi}_Y\left(u; s_1\right)\widehat{\varphi}_Z\left(u; s_2\right)\right|^2 d\mathfrak{w} \\ &= b_T T \int_{\mathbb{R}\times\mathbb{R}} \left|\widehat{\varphi}_{Y,Z}\left(u; s_1, s_2\right) - \widehat{\varphi}_Y\left(u; s_1\right)\widehat{\varphi}_Z\left(u; s_2\right)\right|^2 e^{-\left(|s_1|_1^2+|s_2|_1^2\right)}\, ds_1 ds_2. \end{aligned} \tag{B.1}$$

At first, we concentrate on the difference forming part of the integrand in (B.1), that is

$$\begin{aligned}
&\widehat{\varphi}_{Y,Z}(u;s_1,s_2) - \widehat{\varphi}_Y(u;s_1)\,\widehat{\varphi}_Z(u;s_2) \\
&= \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\,s_1 Y_{t,T} + i\,s_2 Z_{t,T}} \\
&\qquad - \frac{1}{(b_T T)^2}\sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) e^{i\,s_1 Y_{t_1,T}} e^{i\,s_2 Z_{t_2,T}} \\
&=: \mathrm{I} - \mathrm{II}. \qquad \text{(B.2)}
\end{aligned}$$

Since we are interested in the squared absolute value of the above-written difference, we aim to separate real and imaginary part. To do so, we use Euler's formula and get for term I

$$\begin{aligned}
\mathrm{I} &= \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\,s_1 Y_{t,T}} e^{i\,s_2 Z_{t,T}} \\
&= \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(\cos(s_1 Y_{t,T}) + i\sin(s_1 Y_{t,T})\right)\left(\cos(s_2 Z_{t,T}) + i\sin(s_2 Z_{t,T})\right) \\
&= \frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(\cos(s_1 Y_{t,T})\cos(s_2 Z_{t,T}) - \sin(s_1 Y_{t,T})\sin(s_2 Z_{t,T})\right) \\
&\qquad + i\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(\cos(s_1 Y_{t,T})\sin(s_2 Z_{t,T}) + \sin(s_1 Y_{t,T})\cos(s_2 Z_{t,T})\right).
\end{aligned}$$

Now we turn to term II of equation (B.2) and obtain

$$\begin{aligned}
\mathrm{II} &= \frac{1}{(b_T T)^2}\sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad \cdot \left(\cos(s_1 Y_{t_1,T})\cos(s_2 Z_{t_2,T}) - \sin(s_1 Y_{t_1,T})\sin(s_2 Z_{t_2,T})\right) \\
&\qquad + i\frac{1}{(b_T T)^2}\sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad \cdot \left(\cos(s_1 Y_{t_1,T})\sin(s_2 Z_{t_2,T}) + \sin(s_1 Y_{t_1,T})\cos(s_2 Z_{t_2,T})\right)
\end{aligned}$$

in the same manner. Returning to (B.2), we replace I − II by

$$
\begin{aligned}
&\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(s_1 Y_{t,T}\right)\cos\left(s_2 Z_{t,T}\right)-\sin\left(s_1 Y_{t,T}\right)\sin\left(s_2 Z_{t,T}\right)\right)\\
&\qquad\qquad -\frac{1}{\left(b_T T\right)^2}\sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right)K\left(\frac{t_2/T-u}{b_T}\right)\\
&\qquad\qquad \cdot\left(\cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_2,T}\right)-\sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_2,T}\right)\right)\\
&\quad +i\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(s_1 Y_{t,T}\right)\sin\left(s_2 Z_{t,T}\right)+\sin\left(s_1 Y_{t,T}\right)\cos\left(s_2 Z_{t,T}\right)\right)\right.\\
&\qquad\qquad -\frac{1}{\left(b_T T\right)^2}\sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right)K\left(\frac{t_2/T-u}{b_T}\right)\\
&\qquad\qquad \left.\cdot\left(\cos\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_2,T}\right)+\sin\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_2,T}\right)\right)\right)\\
&=: a+ib. \qquad\qquad (\mathrm{B.3})
\end{aligned}
$$

At this point, we got the desired separation and take the squared absolute value into consideration. Due to the form of (B.3), we get

$$|a+ib|^2 = a^2+b^2$$

easily. Returning to the whole integral, equation (B.1) becomes

$$b_T T\,\widehat{\mathfrak{C}}_{Y,Z}(u) = b_T T\int_{\mathbb{R}\times\mathbb{R}} a^2+b^2\,d\mathfrak{w} = b_T T\left(\int_{\mathbb{R}\times\mathbb{R}} a^2\,d\mathfrak{w}+\int_{\mathbb{R}\times\mathbb{R}} b^2\,d\mathfrak{w}\right) \qquad (\mathrm{B.4})$$

thanks to the linearity of the integral. This allows us to examine both summands of (B.4) singly beginning with the first. With help of the binomial theorem, we expand a^2 and obtain

$$
\begin{aligned}
a^2 = &\left(\frac{1}{b_T T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right)\left(\cos\left(s_1 Y_{t,T}\right)\cos\left(s_2 Z_{t,T}\right)-\sin\left(s_1 Y_{t,T}\right)\sin\left(s_2 Z_{t,T}\right)\right)\right)^2\\
&\quad -\frac{2}{\left(b_T T\right)^3}\sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right)K\left(\frac{t_2/T-u}{b_T}\right)K\left(\frac{t_3/T-u}{b_T}\right)\\
&\quad \cdot\left(\cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_1,T}\right)-\sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_1,T}\right)\right)\\
&\quad \cdot\left(\cos\left(s_1 Y_{t_2,T}\right)\cos\left(s_2 Z_{t_3,T}\right)-\sin\left(s_1 Y_{t_2,T}\right)\sin\left(s_2 Z_{t_3,T}\right)\right)\\
&\quad +\left(\frac{1}{\left(b_T T\right)^2}\sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right)K\left(\frac{t_2/T-u}{b_T}\right)\right.\\
&\quad \left.\cdot\left(\cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_2,T}\right)-\sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_2,T}\right)\right)\right)^2\\
=:&\ a_1^2-2\,a_1 a_2+a_2^2.
\end{aligned}
$$

This transfers to the integral as follows:

$$\int_{\mathbb{R}\times\mathbb{R}} a^2\,d\mathfrak{w} = \int_{\mathbb{R}\times\mathbb{R}} a_1^2\,d\mathfrak{w} - 2\int_{\mathbb{R}\times\mathbb{R}} a_1 a_2\,d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} a_2^2\,d\mathfrak{w}. \qquad (\mathrm{B.5})$$

Again, we proceed with the separate investigation of the arisen subterms. We start with the first and use, once more, the linearity of the integral. This leads to

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a_1^2 \, d\mathfrak{w} \\
&= \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_1,T}\right) - \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_1,T}\right)\right) \\
&\qquad\qquad \cdot \left(\cos\left(s_1 Y_{t_2,T}\right)\cos\left(s_2 Z_{t_2,T}\right) - \sin\left(s_1 Y_{t_2,T}\right)\sin\left(s_2 Z_{t_2,T}\right)\right) d\mathfrak{w}. \qquad \text{(B.6)}
\end{aligned}$$

For a moment, we waive the weighted sums including the kernel functions in equation (B.6) and focus on the integral. Expanding of the product yields

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} \left(\cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_1,T}\right) - \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_1,T}\right)\right) \\
&\qquad\qquad \cdot \left(\cos\left(s_1 Y_{t_2,T}\right)\cos\left(s_2 Z_{t_2,T}\right) - \sin\left(s_1 Y_{t_2,T}\right)\sin\left(s_2 Z_{t_2,T}\right)\right) d\mathfrak{w} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_1,T}\right)\cos\left(s_1 Y_{t_2,T}\right)\cos\left(s_2 Z_{t_2,T}\right) e^{-\left(|s_1|_1^2+|s_2|_1^2\right)} \, ds_1 ds_2 \\
&\quad - \int_{\mathbb{R}} \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_1,T}\right)\sin\left(s_1 Y_{t_2,T}\right)\sin\left(s_2 Z_{t_2,T}\right) e^{-\left(|s_1|_1^2+|s_2|_1^2\right)} \, ds_1 ds_2 \\
&\quad - \int_{\mathbb{R}} \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_1,T}\right)\cos\left(s_1 Y_{t_2,T}\right)\cos\left(s_2 Z_{t_2,T}\right) e^{-\left(|s_1|_1^2+|s_2|_1^2\right)} \, ds_1 ds_2 \\
&\qquad + \int_{\mathbb{R}} \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_1,T}\right)\sin\left(s_1 Y_{t_2,T}\right)\sin\left(s_2 Z_{t_2,T}\right) e^{-\left(|s_1|_1^2+|s_2|_1^2\right)} \, ds_1 ds_2. \\
&\hfill \text{(B.7)}
\end{aligned}$$

Now we see that we can divorce the integrals benefiting from the addition theorem appurtenant to the exponential function. Thus, we consider

$$\begin{aligned}
&\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1,T}\right)\cos\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \\
&\quad - \int_{\mathbb{R}} \cos\left(s_2 Z_{t_1,T}\right)\sin\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right)\sin\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \\
&\quad - \int_{\mathbb{R}} \sin\left(s_2 Z_{t_1,T}\right)\cos\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right)\cos\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \\
&\quad + \int_{\mathbb{R}} \sin\left(s_2 Z_{t_1,T}\right)\sin\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \qquad \text{(B.8)}
\end{aligned}$$

in place of (B.7). Since the cosine is a symmetric function, whereas the sine is point-symmetrical, their product inherits the point symmetry. Thus, both the second and third summand of equation (B.8) disappear leaving us with

$$\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1,T}\right) \cos\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right) \cos\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2}\, ds_1$$
$$+ \int_{\mathbb{R}} \sin\left(s_2 Z_{t_1,T}\right) \sin\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right) \sin\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2}\, ds_1 \quad \text{(B.9)}$$

to continue with. Using the cosine's product formula, it holds for the integrand corresponding to the first integral of (B.9)

$$\cos\left(s_2 Z_{t_1,T}\right) \cos\left(s_2 Z_{t_2,T}\right) = \frac{1}{2} \left(\cos\left(s_2 \left(Z_{t_1,T} - Z_{t_2,T}\right)\right) + \cos\left(s_2 \left(Z_{t_1,T} + Z_{t_2,T}\right)\right)\right).$$

Consequently, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1,T}\right) \cos\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2}\, ds_2 \\
&= \frac{1}{2} \int_{\mathbb{R}} \left(\cos\left(s_2 \left(Z_{t_1,T} - Z_{t_2,T}\right)\right) + \cos\left(s_2 \left(Z_{t_1,T} + Z_{t_2,T}\right)\right)\right) e^{-|s_2|_1^2}\, ds_2 \\
&= \frac{1}{2} \left(\int_{\mathbb{R}} \cos\left(s_2 \left(Z_{t_1,T} - Z_{t_2,T}\right)\right) e^{-|s_2|_1^2}\, ds_2 + \int_{\mathbb{R}} \cos\left(s_2 \left(Z_{t_1,T} + Z_{t_2,T}\right)\right) e^{-|s_2|_1^2}\, ds_2\right)
\end{aligned} \quad \text{(B.10)}$$

for the integral itself. Since it holds

$$\int_0^\infty \cos\left(x\beta\right) e^{-x^2}\, dx = \frac{\pi^{1/2}}{2}\, e^{-\beta^2/4} \quad \text{(B.11)}$$

for $\beta \in \mathbb{R}$ as it can be found in Bronstein and Semendjaew (2012), we can compute the first integral in equation (B.10) as follows:

$$\begin{aligned}
\int_{\mathbb{R}} \cos\left(s_2 \left(Z_{t_1,T} - Z_{t_2,T}\right)\right) e^{-|s_2|_1^2}\, ds_2 &= 2 \int_0^\infty \cos\left(s_2 \left(Z_{t_1,T} - Z_{t_2,T}\right)\right) e^{-s_2^2}\, ds_2 \\
&= 2\, \frac{\pi^{1/2}}{2}\, e^{-\frac{\left(Z_{t_1,T} - Z_{t_2,T}\right)^2}{4}} \\
&= \pi^{1/2}\, e^{-\frac{\left(Z_{t_1,T} - Z_{t_2,T}\right)^2}{4}}.
\end{aligned}$$

Similarly, we get

$$\int_{\mathbb{R}} \cos\left(s_2 \left(Z_{t_1,T} + Z_{t_2,T}\right)\right) e^{-|s_2|_1^2}\, ds_2 = \pi^{1/2} e^{-\frac{\left(Z_{t_1,T} + Z_{t_2,T}\right)^2}{4}}$$

for the second integral in (B.10). In combination, equation (B.10) becomes

$$\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1,T}\right) \cos\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2}\, ds_2 = \frac{\pi^{1/2}}{2} \left(e^{-\frac{\left(Z_{t_1,T} - Z_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T} + Z_{t_2,T}\right)^2}{4}}\right). \quad \text{(B.12)}$$

Following the same argumentation regarding the second integral of (B.9), we obtain for the product of the cosine integrals

$$\begin{aligned}
&\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1,T}\right) \cos\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right) \cos\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2}\, ds_1 \\
&= \frac{\pi^{1/2}}{2}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}}\right) \frac{\pi^{1/2}}{2}\left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right) \\
&= \frac{\pi}{4}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}}\right) \cdot \left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right). \qquad \text{(B.13)}
\end{aligned}$$

Now we turn our attention to the third integral of (B.9). As before, we use a trigonometric product formula but this time for the sine. Thus, we get

$$\sin\left(s_2 Z_{t_1,T}\right) \sin\left(s_2 Z_{t_2,T}\right) = \frac{1}{2}\left(\cos\left(s_2\left(Z_{t_1,T} - Z_{t_2,T}\right)\right) - \cos\left(s_2\left(Z_{t_1,T} + Z_{t_2,T}\right)\right)\right) \qquad \text{(B.14)}$$

Since the right-hand side of (B.14) consists only of cosine functions instead of sine ones, we can use equation (B.11) anew and obtain for the integral in question

$$\int_{\mathbb{R}} \sin\left(s_2 Z_{t_1,T}\right) \sin\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2}\, ds_2 = \frac{\pi^{1/2}}{2}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}}\right).$$

Hence, for the product of the sine integrals of equation (B.9) it holds

$$\begin{aligned}
&\int_{\mathbb{R}} \sin\left(s_2 Z_{t_1,T}\right) \sin\left(s_2 Z_{t_2,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right) \sin\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2}\, ds_1 \\
&= \frac{\pi^{1/2}}{2}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}}\right) \frac{\pi^{1/2}}{2}\left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right) \\
&= \frac{\pi}{4}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}}\right) \cdot \left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right). \qquad \text{(B.15)}
\end{aligned}$$

Finally, returning to (B.6) the combination of equations (B.13) and (B.15) leads to

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a_1^2\, d\mathfrak{w} \\
&= \frac{1}{\left(b_T T\right)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T - u}{b_T}\right) K\left(\frac{t_2/T - u}{b_T}\right) \\
&\quad \cdot \left(\frac{\pi}{4}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}}\right) \cdot \left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right)\right. \\
&\qquad \left. + \frac{\pi}{4}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}}\right) \cdot \left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right)\right)
\end{aligned}$$

$$= \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right)$$
$$\cdot \frac{\pi}{2}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right). \quad \text{(B.16)}$$

Now we move on to the second summand of (B.5). By the same arguments as before, we obtain

$$\int_{\mathbb{R}\times\mathbb{R}} a_1 a_2 \, d\mathfrak{w}$$
$$= \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right)$$
$$\cdot \left(\cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_1,T}\right) - \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_1,T}\right)\right)$$
$$\cdot \left(\cos\left(s_1 Y_{t_2,T}\right)\cos\left(s_2 Z_{t_3,T}\right) - \sin\left(s_1 Y_{t_2,T}\right)\sin\left(s_2 Z_{t_3,T}\right)\right) d\mathfrak{w}$$
$$= \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right)$$
$$\cdot \left(\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1,T}\right)\cos\left(s_2 Z_{t_3,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1\right.$$
$$\left. + \int_{\mathbb{R}} \sin\left(s_2 Z_{t_1,T}\right)\sin\left(s_2 Z_{t_3,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_1 Y_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1\right). \quad \text{(B.17)}$$

Now we see that, again, we have products of integrals with either the sine or the cosine function. Thus, we keep following the lines corresponding to a_1^2 and get

$$\int_{\mathbb{R}\times\mathbb{R}} a_1 a_2 \, d\mathfrak{w}$$
$$= \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right)$$
$$\cdot \frac{\pi}{2}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right) \quad \text{(B.18)}$$

in preference to equation (B.17). Finally, we look at the last summand of (B.5). Completely analogue to the previous two examinations, we obtain

$$\int_{\mathbb{R}\times\mathbb{R}} a_2^2 \, d\mathfrak{w}$$
$$= \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{(b_T T)^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right)$$
$$\cdot \left(\cos\left(s_1 Y_{t_1,T}\right)\cos\left(s_2 Z_{t_2,T}\right) - \sin\left(s_1 Y_{t_1,T}\right)\sin\left(s_2 Z_{t_2,T}\right)\right)$$
$$\cdot \left(\cos\left(s_1 Y_{t_3,T}\right)\cos\left(s_2 Z_{t_4,T}\right) - \sin\left(s_1 Y_{t_3,T}\right)\sin\left(s_2 Z_{t_4,T}\right)\right) d\mathfrak{w}$$

$$
\begin{aligned}
&= \frac{1}{(b_T T)^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\quad \cdot \left(\int_{\mathbb{R}} \cos\left(s_2 Z_{t_2,T}\right) \cos\left(s_2 Z_{t_4,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right) \cos\left(s_1 Y_{t_3,T}\right) e^{-|s_1|_1^2}\, ds_1 \right. \\
&\qquad \left. + \int_{\mathbb{R}} \sin\left(s_2 Z_{t_2,T}\right) \sin\left(s_2 Z_{t_4,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right) \sin\left(s_1 Y_{t_3,T}\right) e^{-|s_1|_1^2}\, ds_1 \right) \\
&= \frac{1}{(b_T T)^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad \cdot \frac{\pi}{2} \left(e^{-\frac{\left(Z_{t_2,T}-Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_3,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_2,T}+Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_3,T}\right)^2}{4}} \right).
\end{aligned} \tag{B.19}
$$

Before we combine all of these results, we turn our attention to the imaginary part of the original difference. Afterwards, we will see that we can conflate both th real and imaginary part in a convenient way. Therefore, we go back to equation (B.4) for now and expand b^2 as follows:

$$
\begin{aligned}
b^2 &= \left(\frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(\cos\left(s_1 Y_{t,T}\right) \sin\left(s_2 Z_{t,T}\right) + \sin\left(s_1 Y_{t,T}\right) \cos\left(s_2 Z_{t,T}\right)\right) \right)^2 \\
&\quad - \frac{2}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad \cdot \left(\cos\left(s_1 Y_{t_1,T}\right) \sin\left(s_2 Z_{t_1,T}\right) + \sin\left(s_1 Y_{t_1,T}\right) \cos\left(s_2 Z_{t_1,T}\right)\right) \\
&\qquad \cdot \left(\cos\left(s_1 Y_{t_2,T}\right) \sin\left(s_2 Z_{t_3,T}\right) + \sin\left(s_1 Y_{t_2,T}\right) \cos\left(s_2 Z_{t_3,T}\right)\right) \\
&\qquad + \left(\frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \right. \\
&\qquad\qquad \left. \cdot \left(\cos\left(s_1 Y_{t_1,T}\right) \sin\left(s_2 Z_{t_2,T}\right) + \sin\left(s_1 Y_{t_1,T}\right) \cos\left(s_2 Z_{t_2,T}\right)\right)\right)^2 \\
&=: b_1^2 - 2\, b_1 b_2 + b_2^2.
\end{aligned}
$$

Hence, it holds for the integral

$$
\int_{\mathbb{R}\times\mathbb{R}} b^2\, d\mathfrak{w} = \int_{\mathbb{R}\times\mathbb{R}} b_1^2\, d\mathfrak{w} - 2 \int_{\mathbb{R}\times\mathbb{R}} b_1 b_2\, d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} b_2^2\, d\mathfrak{w} \tag{B.20}
$$

similar to (B.5). Because of the linearity of the integral and the sine's point symmetry, we obtain for the first summand of equation (B.20)

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} b_1^2 \, d\mathfrak{w} \\
&= \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\quad \cdot \left(\int_{\mathbb{R}} \sin(s_2 Z_{t_1,T}) \sin(s_2 Z_{t_2,T}) \, e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos(s_1 Y_{t_1,T}) \cos(s_1 Y_{t_2,T}) \, e^{-|s_1|_1^2} \, ds_1 \right. \\
&\qquad \left. + \int_{\mathbb{R}} \cos(s_2 Z_{t_1,T}) \cos(s_2 Z_{t_2,T}) \, e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin(s_1 Y_{t_1,T}) \sin(s_1 Y_{t_2,T}) \, e^{-|s_1|_1^2} \, ds_1 \right).
\end{aligned} \tag{B.21}$$

Again, we apply the trigonometric product formulas and get

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} b_1^2 \, d\mathfrak{w} \\
&= \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\quad \cdot \left(\frac{\pi}{4} \left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}} \right) \cdot \left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}} \right) \right. \\
&\qquad \left. + \frac{\pi}{4} \left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}} \right) \cdot \left(e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}} \right) \right) \\
&= \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad \cdot \frac{\pi}{2} \left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}} \right)
\end{aligned} \tag{B.22}$$

in place of (B.21). Next, we examine the second summand of (B.20). Analogously to the first, we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} b_1 b_2 \, d\mathfrak{w} \\
&= \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\quad \cdot \left(\int_{\mathbb{R}} \sin(s_2 Z_{t_1,T}) \sin(s_2 Z_{t_3,T}) \, e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos(s_1 Y_{t_1,T}) \cos(s_1 Y_{t_2,T}) \, e^{-|s_1|_1^2} \, ds_1 \right. \\
&\qquad \left. + \int_{\mathbb{R}} \cos(s_2 Z_{t_1,T}) \cos(s_2 Z_{t_3,T}) \, e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin(s_1 Y_{t_1,T}) \sin(s_1 Y_{t_2,T}) \, e^{-|s_1|_1^2} \, ds_1 \right)
\end{aligned}$$

$$= \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \cdot \frac{\pi}{2}\left(e^{-\frac{\left(Z_{t_1,T}-Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}}\right), \tag{B.23}$$

and finally, for the third summand of equation (B.20), it holds

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} b_2^2 \, d\mathfrak{w} \\
&= \frac{1}{(b_T T)^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\quad \cdot \left(\int_{\mathbb{R}} \sin\left(s_2 Z_{t_2,T}\right) \sin\left(s_2 Z_{t_4,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1,T}\right) \cos\left(s_1 Y_{t_3,T}\right) e^{-|s_1|_1^2} \, ds_1 \right. \\
&\qquad \left. + \int_{\mathbb{R}} \cos\left(s_2 Z_{t_2,T}\right) \cos\left(s_2 Z_{t_4,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1,T}\right) \sin\left(s_1 Y_{t_3,T}\right) e^{-|s_1|_1^2} \, ds_1 \right) \\
&= \frac{1}{(b_T T)^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad \cdot \frac{\pi}{2}\left(e^{-\frac{\left(Z_{t_2,T}-Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_3,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_2,T}+Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_3,T}\right)^2}{4}}\right).
\end{aligned} \tag{B.24}$$

Now it is time to bring our results together. Instead of combining both the integrals belonging to the real part and those appurtenant to the imaginary part singly, we form pairs first. To put a finer point on that, we go back to equation (B.4). With the help of (B.5) and (B.20), we restructure $\widehat{\mathfrak{C}}_{Y,Z}(u)$, to wit

$$\begin{aligned}
&\widehat{\mathfrak{C}}_{Y,Z}(u) \\
&= \int_{\mathbb{R}\times\mathbb{R}} a_1^2 \, d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} b_1^2 \, d\mathfrak{w} - 2\left(\int_{\mathbb{R}\times\mathbb{R}} a_1 a_2 \, d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} b_1 b_2 \, d\mathfrak{w}\right) \\
&\qquad + \int_{\mathbb{R}\times\mathbb{R}} a_2^2 \, d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} b_2^2 \, d\mathfrak{w} \\
&=: c_1 - 2\, c_2 + c_3.
\end{aligned} \tag{B.25}$$

Beginning with c_1, we look at the three newly defined subterms individually. Regarding equations (B.16) and (B.22), we have

$$
\begin{aligned}
c_1 = \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
\cdot \frac{\pi}{2}\Bigg(e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}} \\
+ e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}} \Bigg) \\
= \frac{\pi}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}}.
\end{aligned}
\tag{B.26}
$$

Moving on to c_2, it holds

$$
\begin{aligned}
c_2 = \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
\cdot \frac{\pi}{2}\Bigg(e^{-\frac{\left(Z_{t_1,T}-Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_1,T}+Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}} \\
+ e^{-\frac{\left(Z_{t_1,T}-Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_1,T}+Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_2,T}\right)^2}{4}} \Bigg) \\
= \frac{\pi}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
\cdot e^{-\frac{\left(Z_{t_1,T}-Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}}
\end{aligned}
\tag{B.27}
$$

due to equations (B.18) and (B.23). For the last subterm of (B.25), we get using equations (B.19) and (B.24)

$$
\begin{aligned}
c_3 = \frac{1}{(b_T T)^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
\cdot \frac{\pi}{2}\Bigg(e^{-\frac{\left(Z_{t_2,T}-Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_3,T}\right)^2}{4}} + e^{-\frac{\left(Z_{t_2,T}+Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_3,T}\right)^2}{4}} \\
+ e^{-\frac{\left(Z_{t_2,T}-Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_3,T}\right)^2}{4}} - e^{-\frac{\left(Z_{t_2,T}+Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}+Y_{t_3,T}\right)^2}{4}} \Bigg) \\
= \frac{\pi}{(b_T T)^4} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
\cdot e^{-\frac{\left(Z_{t_2,T}-Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_3,T}\right)^2}{4}}.
\end{aligned}
\tag{B.28}
$$

With equations (B.26), (B.27) and (B.28) in mind, we return now to equation (B.25) and take the prefactor $b_T T$ back into consideration. This leads to

$$
\begin{aligned}
& b_T T\, \widehat{\mathfrak{C}}_{Y,Z}(u) \\
&= \pi \Bigg(\frac{1}{b_T T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) e^{-\frac{\left(Z_{t_1,T}-Z_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} \\
&\qquad - \frac{2}{(b_T T)^2} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z_{t_1,T}-Z_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_2,T}\right)^2}{4}} \\
&\quad + \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z_{t_2,T}-Z_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1,T}-Y_{t_3,T}\right)^2}{4}} \Bigg).
\end{aligned}
$$

This finishes the first term transformation. Now we enter the bootstrap world and take a closer look at $b_T T\, \widehat{\mathfrak{C}}^{\star}_{Y,Z}(u)$. As it holds

$$
\begin{aligned}
\widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) = \int_{\mathbb{R}\times\mathbb{R}} \Big| & \widehat{\varphi}^{\star}_{Y,Z}(u; s_1, s_2) - E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u; s_1, s_2) + E^{\star}\widehat{\varphi}^{\star}_{Y}(u; s_1)\; E^{\star}\widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) \\
& - \widehat{\varphi}^{\star}_{Y}(u; s_1)\; \widehat{\varphi}^{\star}_{Z}(u; s_2) \Big|^2 \, d\mathfrak{w},
\end{aligned}
$$

we start by rearranging the summands of the integrand in the following way:

$$
\begin{aligned}
\Big| & \widehat{\varphi}^{\star}_{Y,Z}(u; s_1, s_2) - \widehat{\varphi}^{\star}_{Y}(u; s_1)\; \widehat{\varphi}^{\star}_{Z}(u; s_2) \\
& - \left(E^{\star}\widehat{\varphi}^{\star}_{Y,Z}(u; s_1, s_2) - E^{\star}\widehat{\varphi}^{\star}_{Y}(u; s_1)\; E^{\star}\widehat{\varphi}^{\star}_{Z}(u; \underline{s}_2) \right) \Big|^2 .
\end{aligned}
\tag{B.29}
$$

Next, we define

$$
\begin{aligned}
a^{\star} := {} & \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(\cos\left(s_1 Y^{\star}_{t,T}\right) \cos\left(s_2 Z^{\star}_{t,T}\right) - \sin\left(s_1 Y^{\star}_{t,T}\right) \sin\left(s_2 Z^{\star}_{t,T}\right) \right) \\
& - \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
& \qquad \cdot \left(\cos\left(s_1 Y^{\star}_{t_1,T}\right) \cos\left(s_2 Z^{\star}_{t_2,T}\right) - \sin\left(s_1 Y^{\star}_{t_1,T}\right) \sin\left(s_2 Z^{\star}_{t_2,T}\right) \right)
\end{aligned}
$$

and

$$
\begin{aligned}
b^\star := \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(\cos\left(s_1 Y_{t,T}^\star\right) \sin\left(s_2 Z_{t,T}^\star\right) + \sin\left(s_1 Y_{t,T}^\star\right) \cos\left(s_2 Z_{t,T}^\star\right)\right) \\
- \frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
\cdot \left(\cos\left(s_1 Y_{t_1,T}^\star\right) \sin\left(s_2 Z_{t_2,T}^\star\right) + \sin\left(s_1 Y_{t_1,T}^\star\right) \cos\left(s_2 Z_{t_2,T}^\star\right)\right)
\end{aligned}
$$

together with

$$
\begin{aligned}
\breve{a} := \frac{1}{2\,TD_T+1} \frac{1}{b_T T} \sum_{r=-TD_T}^{TD_T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \\
\cdot \left(\cos\left(s_1 Y_{t+r,T}\right) \cos\left(s_2 Z_{t+r,T}\right) - \sin\left(s_1 Y_{t+r,T}\right) \sin\left(s_2 Z_{t+r,T}\right)\right) \\
- \frac{1}{(2\,TD_T+1)^2} \frac{1}{(b_T T)^2} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
\cdot \left(\cos\left(s_1 Y_{t_1+r_1,T}\right) \cos\left(s_2 Z_{t_2+r_2,T}\right) - \sin\left(s_1 Y_{t_1+r_1,T}\right) \sin\left(s_2 Z_{t_2+r_2,T}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\breve{b} := \frac{1}{2\,TD_T+1} \frac{1}{b_T T} \sum_{r=-TD_T}^{TD_T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \\
\cdot \left(\cos\left(s_1 Y_{t+r,T}\right) \sin\left(s_2 Z_{t+r,T}\right) + \sin\left(s_1 Y_{t+r,T}\right) \cos\left(s_2 Z_{t+r,T}\right)\right) \\
- \frac{1}{(2\,TD_T+1)^2} \frac{1}{(b_T T)^2} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
\cdot \left(\cos\left(s_1 Y_{t_1+r_1,T}\right) \sin\left(s_2 Z_{t_2+r_2,T}\right) + \sin\left(s_1 Y_{t_1+r_1,T}\right) \cos\left(s_2 Z_{t_2+r_2,T}\right)\right).
\end{aligned}
$$

Thereby, (B.29) becomes

$$
\left|a^\star + ib^\star - \left(\breve{a} + i\breve{b}\right)\right|^2 = \left|a^\star - \breve{a} + i\left(b^\star - \breve{b}\right)\right|^2 = \left(a^\star - \breve{a}\right)^2 + \left(b^\star - \breve{b}\right)^2
$$

analogously to the real world counterpart. Because of

$$
\left(a^\star - \breve{a}\right)^2 = a^{\star 2} - 2\,a^\star\breve{a} + \breve{a}^2 \qquad \text{and} \qquad \left(b^\star - \breve{b}\right)^2 = b^{\star 2} - 2\,b^\star\breve{b} + \breve{b}^2,
$$

we can rewrite $\widehat{\mathfrak{C}}^\star_{Y,Z}(u)$ as follows:

$$
\widehat{\mathfrak{C}}^\star_{Y,Z}(u) = \int_{\mathbb{R}\times\mathbb{R}} a^{\star 2} + b^{\star 2}\, d\mathfrak{w} - 2\int_{\mathbb{R}\times\mathbb{R}} a^\star\breve{a} + b^\star\breve{b}\, d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} \breve{a}^2 + \breve{b}^2\, d\mathfrak{w}. \tag{B.30}
$$

Now we continue by investigating the three integrals of (B.30) one by one. As $a^\star$ and $b^\star$ are the bootstrap versions of a and b, respectively, we can transfer the results of the real world to get

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a^{\star 2} + b^{\star 2}\, d\mathfrak{w} \\
&= \pi \Bigg(\frac{1}{(b_T T)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) e^{-\frac{\left(Z^\star_{t_1,T}-Z^\star_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y^\star_{t_1,T}-Y^\star_{t_2,T}\right)^2}{4}} \\
&\qquad - \frac{2}{(b_T T)^3} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^\star_{t_1,T}-Z^\star_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y^\star_{t_1,T}-Y^\star_{t_2,T}\right)^2}{4}} \\
&\quad + \frac{1}{(b_T T)^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^\star_{t_2,T}-Z^\star_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y^\star_{t_1,T}-Y^\star_{t_3,T}\right)^2}{4}} \Bigg)
\end{aligned} \tag{B.31}$$

without further calculations. The same principle works for the third integral on the right-hand side of equation (B.30). However, the addition of the bootstrap expectation involves some modifications. These include additional sums and prefactors as well as index shifts. This leads to

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} \breve{a}^2 + \breve{b}^2\, d\mathfrak{w} \\
&= \pi \Bigg(\frac{1}{(2TD_T+1)^2} \frac{1}{(b_T T)^2} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad \cdot e^{-\frac{\left(Z_{t_1+r_1,T}-Z_{t_2+r_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1+r_1,T}-Y_{t_2+r_2,T}\right)^2}{4}} \\
&\qquad - \frac{2}{(2TD_T+1)^3} \frac{1}{(b_T T)^3} \sum_{r_1,r_2,r_3=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} \\
&\qquad K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad \cdot e^{-\frac{\left(Z_{t_1+r_1,T}-Z_{t_3+r_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1+r_1,T}-Y_{t_2+r_2,T}\right)^2}{4}} \\
&\qquad + \frac{1}{(2TD_T+1)^4} \frac{1}{(b_T T)^4} \sum_{r_1,r_2,r_3,r_4=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3,t_4=1}^{T} \\
&\qquad K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z_{t_2+r_2,T}-Z_{t_4+r_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1+r_1,T}-Y_{t_3+r_3,T}\right)^2}{4}} \Bigg).
\end{aligned} \tag{B.32}$$

Now we move on to the last summand in (B.30). Since we have to deal with a mixture of bootstrap variables and expectations, a deeper examination is necessary in this case. Firstly, we divide the integral as follows:

$$\int_{\mathbb{R}\times\mathbb{R}} a^\star \breve{a} + b^\star \breve{b}\, d\mathfrak{w} = \int_{\mathbb{R}\times\mathbb{R}} a^\star \breve{a}\, d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} b^\star \breve{b}\, d\mathfrak{w}. \tag{B.33}$$

We will confine ourselves to the detailed investigation of the fist new integral of the right-hand side of (B.33) because we have seen in the real world's calculations that the second integral can be computed in a quite similar way. In complete analogy to the real world counterpart, we define

$$a_1^\star := \frac{1}{b_T T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \left(\cos\left(s_1 Y_{t,T}^\star\right) \cos\left(s_2 Z_{t,T}^\star\right) - \sin\left(s_1 Y_{t,T}^\star\right) \sin\left(s_2 Z_{t,T}^\star\right)\right)$$

and

$$a_2^\star := \frac{1}{\left(b_T T\right)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \cdot \left(\cos\left(s_1 Y_{t_1,T}^\star\right) \cos\left(s_2 Z_{t_2,T}^\star\right) - \sin\left(s_1 Y_{t_1,T}^\star\right) \sin\left(s_2 Z_{t_2,T}^\star\right)\right)$$

as well as

$$\breve{a}_1 := \frac{1}{2TD_T+1} \frac{1}{b_T T} \sum_{r=-TD_T}^{TD_T} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) \cdot \left(\cos\left(s_1 Y_{t+r,T}\right) \cos\left(s_2 Z_{t+r,T}\right) - \sin\left(s_1 Y_{t+r,T}\right) \sin\left(s_2 Z_{t+r,T}\right)\right)$$

and

$$\breve{a}_2 := \frac{1}{\left(2TD_T\right)^2} \frac{1}{\left(b_T T\right)^2} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \cdot \left(\cos\left(s_1 Y_{t_1+r_1,T}\right) \cos\left(s_2 Z_{t_2+r_2,T}\right) - \sin\left(s_1 Y_{t_1+r_1,T}\right) \sin\left(s_2 Z_{t_2+r_2,T}\right)\right).$$

Thereby, it holds both $a^\star = a_1^\star - a_2^\star$ and $\breve{a} = \breve{a}_1 - \breve{a}_2$, which means for the product

$$a^\star \breve{a} = a_1^\star \breve{a}_1 - a_1^\star \breve{a}_2 - a_2^\star \breve{a}_1 + a_2^\star \breve{a}_2.$$

Transferred to the integral, we get

$$\int_{\mathbb{R}\times\mathbb{R}} a^\star \breve{a}\, d\mathfrak{w} = \int_{\mathbb{R}\times\mathbb{R}} a_1^\star \breve{a}_1\, d\mathfrak{w} - \int_{\mathbb{R}\times\mathbb{R}} a_1^\star \breve{a}_2\, d\mathfrak{w} - \int_{\mathbb{R}\times\mathbb{R}} a_2^\star \breve{a}_1\, d\mathfrak{w} + \int_{\mathbb{R}\times\mathbb{R}} a_2^\star \breve{a}_2\, d\mathfrak{w}. \tag{B.34}$$

We proceed with the computation of these integrals one after another. As we have four instead of three new integrals to examine now, the difference between the former cases becomes clear. However, we can use the same formulas as before to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a_1^\star \, \breve{a}_1 \, d\mathfrak{w} \\
&= \frac{1}{2\,TD_T+1} \frac{1}{(b_T T)^2} \sum_{r_1=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\quad \cdot \Bigg(\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1+r_1,T}\right) \cos\left(s_2 Z^\star_{t_2,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1+r_1,T}\right) \cos\left(s_1 Y^\star_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \\
&\qquad + \int_{\mathbb{R}} \sin\left(s_2 Z_{t_1+r_1,T}\right) \sin\left(s_2 Z^\star_{t_2,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1+r_1,T}\right) \sin\left(s_1 Y^\star_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \Bigg)
\end{aligned}
$$

for the first integral of the right-hand side of equation (B.34). In the same way, we get

$$
\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a_1^\star \, \breve{a}_2 \, d\mathfrak{w} \\
&= \frac{1}{(2\,TD_T+1)^2} \frac{1}{(b_T T)^3} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\quad \cdot \Bigg(\int_{\mathbb{R}} \cos\left(s_2 Z_{t_2+r_2,T}\right) \cos\left(s_2 Z^\star_{t_3,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1+r_1,T}\right) \cos\left(s_1 Y^\star_{t_3,T}\right) e^{-|s_1|_1^2} \, ds_1 \\
&\qquad + \int_{\mathbb{R}} \sin\left(s_2 Z_{t_2+r_2,T}\right) \sin\left(s_2 Z^\star_{t_3,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1+r_1,T}\right) \sin\left(s_1 Y^\star_{t_3,T}\right) e^{-|s_1|_1^2} \, ds_1 \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a_2^\star \, \breve{a}_1 \, d\mathfrak{w} \\
&= \frac{1}{2\,TD_T+1} \frac{1}{(b_T T)^3} \sum_{r_1=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\quad \cdot \Bigg(\int_{\mathbb{R}} \cos\left(s_2 Z_{t_1+r_1,T}\right) \cos\left(s_2 Z^\star_{t_3,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1+r_1,T}\right) \cos\left(s_1 Y^\star_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \\
&\qquad + \int_{\mathbb{R}} \sin\left(s_2 Z_{t_1+r_1,T}\right) \sin\left(s_2 Z^\star_{t_3,T}\right) e^{-|s_2|_1^2} \, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1+r_1,T}\right) \sin\left(s_1 Y^\star_{t_2,T}\right) e^{-|s_1|_1^2} \, ds_1 \Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a_2^\star\, \breve{a}_2 \, d\mathfrak{w} \\
&= \frac{1}{(2\,TD_T+1)^2} \frac{1}{(b_T T)^4} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3,t_4=1}^{T} \\
&\qquad K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\quad \cdot \Bigg(\int_{\mathbb{R}} \cos\left(s_2 Z_{t_2+r_2,T}\right) \cos\left(s_2 Z^\star_{t_4,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \cos\left(s_1 Y_{t_1+r_1,T}\right) \cos\left(s_1 Y^\star_{t_3,T}\right) e^{-|s_1|_1^2}\, ds_1 \\
&\qquad + \int_{\mathbb{R}} \sin\left(s_2 Z_{t_2+r_2,T}\right) \sin\left(s_2 Z^\star_{t_4,T}\right) e^{-|s_2|_1^2}\, ds_2 \int_{\mathbb{R}} \sin\left(s_1 Y_{t_1+r_1,T}\right) \sin\left(s_1 Y^\star_{t_3,T}\right) e^{-|s_1|_1^2}\, ds_1 \Bigg)
\end{aligned}
$$

for the remaining integrals of (B.34). Combined with the corresponding results for the second integral of the right-hand side of equation (B.33), this leads to

$$
\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} a^\star \breve{a} + b^\star \breve{b} \, d\mathfrak{w} \\
&= \pi \Bigg(\frac{1}{2\,TD_T+1} \frac{1}{(b_T T)^2} \sum_{r_1=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^\star_{t_2,T}-Z_{t_1+r_1,T}\right)^2}{4}} e^{-\frac{\left(Y^\star_{t_2,T}-Y_{t_1+r_1,T}\right)^2}{4}} \\
&- \frac{1}{(2\,TD_T+1)^2} \frac{1}{(b_T T)^3} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^\star_{t_3,T}-Z_{t_2+r_2,T}\right)^2}{4}} e^{-\frac{\left(Y^\star_{t_3,T}-Y_{t_1+r_1,T}\right)^2}{4}} \\
&\quad - \frac{1}{2\,TD_T+1} \frac{1}{(b_T T)^3} \sum_{r_1=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^\star_{t_3,T}-Z_{t_1+r_1,T}\right)^2}{4}} e^{-\frac{\left(Y^\star_{t_2,T}-Y_{t_1+r_1,T}\right)^2}{4}} \\
&\qquad\qquad + \frac{1}{(2\,TD_T+1)^2} \frac{1}{(b_T T)^4} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3,t_4=1}^{T} \\
&\qquad K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad\qquad\qquad \cdot e^{-\frac{\left(Z^\star_{t_4,T}-Z_{t_2+r_2,T}\right)^2}{4}} e^{-\frac{\left(Y^\star_{t_3,T}-Y_{t_1+r_1,T}\right)^2}{4}} \Bigg).
\end{aligned} \tag{B.35}
$$

Joining equations (B.31), (B.32) and (B.35), we obtain finally

$$
\begin{aligned}
& b_T T\, \widehat{\mathfrak{C}}^{\star}_{Y,Z}(u) \\
&= \pi \Bigg(\frac{1}{b_T T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) e^{-\frac{\left(Z^{\star}_{t_1,T}-Z^{\star}_{t_2,T}\right)^2}{4}} e^{-\frac{\left(Y^{\star}_{t_1,T}-Y^{\star}_{t_2,T}\right)^2}{4}} \\
&\qquad - \frac{2}{(b_T T)^2} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^{\star}_{t_1,T}-Z^{\star}_{t_3,T}\right)^2}{4}} e^{-\frac{\left(Y^{\star}_{t_1,T}-Y^{\star}_{t_2,T}\right)^2}{4}} \\
&\quad + \frac{1}{(b_T T)^3} \sum_{t_1,t_2,t_3,t_4=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^{\star}_{t_2,T}-Z^{\star}_{t_4,T}\right)^2}{4}} e^{-\frac{\left(Y^{\star}_{t_1,T}-Y^{\star}_{t_3,T}\right)^2}{4}} \\
&- \frac{2}{2TD_T+1} \frac{1}{b_T T} \sum_{r_1=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^{\star}_{t_2,T}-Z_{t_1+r_1,T}\right)^2}{4}} e^{-\frac{\left(Y^{\star}_{t_2,T}-Y_{t_1+r_1,T}\right)^2}{4}} \\
&+ \frac{2}{(2TD_T+1)^2} \frac{1}{(b_T T)^2} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^{\star}_{t_3,T}-Z_{t_2+r_2,T}\right)^2}{4}} e^{-\frac{\left(Y^{\star}_{t_3,T}-Y_{t_1+r_1,T}\right)^2}{4}} \\
&\quad + \frac{2}{2TD_T+1} \frac{1}{(b_T T)^2} \sum_{r_1=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^{\star}_{t_3,T}-Z_{t_1+r_1,T}\right)^2}{4}} e^{-\frac{\left(Y^{\star}_{t_2,T}-Y_{t_1+r_1,T}\right)^2}{4}} \\
&\qquad - \frac{2}{(2TD_T+1)^2} \frac{1}{(b_T T)^3} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3,t_4=1}^{T} \\
&\qquad \cdot K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad\qquad \cdot e^{-\frac{\left(Z^{\star}_{t_4,T}-Z_{t_2+r_2,T}\right)^2}{4}} e^{-\frac{\left(Y^{\star}_{t_3,T}-Y_{t_1+r_1,T}\right)^2}{4}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{(2TD_T+1)^2} \frac{1}{b_T T} \sum_{r_1,r_2=-TD_T}^{TD_T} \sum_{t_1,t_2=1}^{T} K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) \\
&\qquad \cdot e^{-\frac{\left(Z_{t_1+r_1,T}-Z_{t_2+r_2,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1+r_1,T}-Y_{t_2+r_2,T}\right)^2}{4}} \\
&- \frac{2}{(2TD_T+1)^3} \frac{1}{(b_T T)^2} \sum_{r_1,r_2,r_3=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3=1}^{T} \\
&\qquad K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) \\
&\qquad \cdot e^{-\frac{\left(Z_{t_1+r_1,T}-Z_{t_3+r_3,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1+r_1,T}-Y_{t_2+r_2,T}\right)^2}{4}} \\
&+ \frac{1}{(2TD_T+1)^4} \frac{1}{(b_T T)^3} \sum_{r_1,r_2,r_3,r_4=-TD_T}^{TD_T} \sum_{t_1,t_2,t_3,t_4=1}^{T} \\
&\qquad K\left(\frac{t_1/T-u}{b_T}\right) K\left(\frac{t_2/T-u}{b_T}\right) K\left(\frac{t_3/T-u}{b_T}\right) K\left(\frac{t_4/T-u}{b_T}\right) \\
&\qquad \left. \cdot e^{-\frac{\left(Z_{t_2+r_2,T}-Z_{t_4+r_4,T}\right)^2}{4}} e^{-\frac{\left(Y_{t_1+r_1,T}-Y_{t_3+r_3,T}\right)^2}{4}} \right) .
\end{aligned}
$$

Clearly, it is possible to encapsulate some of the sums above, but since this does not contribute to a neater form, we refrain from doing that.

B.2.2. Codes Belonging to Chapter 7

In this subsection, we return to the codes which benefit from the calculations made in the previous subsection.

To begin with, we look at the independent case. As in Subsection B.1.2, we choose one exemplary parameter combination instead of listing all codes. Whereas the values for δ_1, δ_2 and δ_3 are fixed at $0.14, 0.8$ and 0.41, respectively, we have the option to choose values for T and u, which will be 5000 and 0.6, respectively. Moreover, in this scenario we deal with different repetition numbers. In the code presented below, we consider $N = 200$ and $N_{BS} = 500$. De novo, we adhere to $s = 0.6$.

```
set.seed(1)

r<-function(x){0.9*sin(2*pi*x)}

K<-function(x){ if(x>1 | x<(-1))
  0
  else
    (3/4)*(1-x^2)
}

library(stabledist)
library(purrr)

delta1<-0.14
```

```
delta2<-0.8
delta3<-0.41
alpha<-1.5
gamma_roh<-0.5
mu<-0
beta<-0

gamma<-(gamma_roh)^(1/alpha)

T<-5000
b<-T^(-delta3)
u<-0.6

Karg<-function(x){(x/T-u)/b}
Kc<-ceiling(T*(u-b))
Kf<-floor(T*(u+b))
Kc1<-Kc-1
tvec<-seq(Kc,Kf,1)
len<-length(tvec)
Argvec<-sapply(tvec,FUN=Karg)
Kvec<-sapply(Argvec,FUN=K)

KK<-Kvec%*%t(Kvec)
KKK<-sapply(Kvec,function(y){KK*y})

ef<-function(A,C){exp(-((A-C)^2)/4)}

L<-floor(T^delta1)
F_in<-floor(T^(1-delta2))
F<-2*F_in+1

BS_u<-ceiling((Kc-F_in)/L)
BS_o<-ceiling((Kf+F_in)/L)
BS_g<-BS_o-BS_u+1

N<-200
N_BS<-500

Isum<-rep(0,N)
Isum1<-rep(0,N)
Isum2<-rep(0,N)
Isum3a<-rep(0,N)
Isum3b<-rep(0,N)

Isum_Estern<-rep(0,N)
Isum1_Estern<-rep(0,N)
Isum2_Estern<-rep(0,N)
Isum3a_Estern<-rep(0,N)
Isum3b_Estern<-rep(0,N)

Y_stern<-mat.or.vec(T,N_BS)
Z_stern<-mat.or.vec(T,N_BS)

Yab_stern<-rep(0,len)
Zab_stern<-rep(0,len)

Isum<-double(N)
qo_Isum<-double(N)
Cov_Isum<-0

for(m in 1:N)
{
  Y<-double(T)
  Z<-double(T)

  epsY<-rstable(T,alpha,beta,gamma,mu)
  epsZ<-rstable(T,alpha,beta,gamma,mu)

  Y[1]<-r(1/T)*rstable(1,alpha,beta,gamma,mu)+epsY[1]
  Z[1]<-r(1/T)*rstable(1,alpha,beta,gamma,mu)+epsZ[1]
```

```
for(i in 2:T)
{
  Y[i]<-r(i/T)*Y[i-1]+epsY[i]
  Z[i]<-r(i/T)*Z[i-1]+epsZ[i]
}

Yab<-Y[tvec]
Zab<-Z[tvec]

efYab<-sapply(Yab,function(x){sapply(Yab,function(y){ef(x,y)})})
efZab<-sapply(Zab,function(x){sapply(Zab,function(y){ef(x,y)})})

efPab<-efYab*efZab

Isum1[m]<-sum(KK*efPab)
Isum3a[m]<-sum(KK*efZab)
Isum3b[m]<-sum(KK*efYab)

Pabmat<-sapply(c(1:len),function(y){efZab[,y]%*%t(efYab[,y])})

Isum2[m]<-sum(KKK*Pabmat)

Isum[m]<-pi*(Isum1[m]/((b*T))-2*Isum2[m]/((b*T)^2)+(Isum3a[m]*Isum3b[m])/((b*T)^3))

efYg<-matrix(0,len,len)
efZg<-matrix(0,len,len)
efPg<-matrix(0,len,len)
Pmatg<-matrix(0,len^2,len)

for(r1 in -F_in:F_in)
{
  Yr1<-Y[(Kc1+r1+1):(Kc1+r1+len)]
  Zr1<-Z[(Kc1+r1+1):(Kc1+r1+len)]

  for(r2 in -F_in:F_in)
  {
    Yr2<-Y[(Kc1+r2+1):(Kc1+r2+len)]
    Zr2<-Z[(Kc1+r2+1):(Kc1+r2+len)]

    efY<-sapply(Yr1,function(x){sapply(Yr2,function(y){ef(x,y)})})
    efZ<-sapply(Zr1,function(x){sapply(Zr2,function(y){ef(x,y)})})

    efP<-efY*efZ

    efPg<-efPg+efP
    efYg<-efYg+efY
    efZg<-efZg+efZ

    for(r3 in -F_in:F_in)
    {
      Yr3<-Y[(Kc1+r3+1):(Kc1+r3+len)]

      efYr3<-sapply(Yr1,function(x){sapply(Yr3,function(y){ef(x,y)})})

      Pmat<-sapply(c(1:len),function(y){efZ[,y]%*%t(efYr3[,y])})

      Pmatg<-Pmatg+Pmat
    }
  }
}

Isum1_Estern[m]<-sum(KK*efPg)
Isum3a_Estern[m]<-sum(KK*efZg)
Isum3b_Estern[m]<-sum(KK*efYg)
Isum2_Estern[m]<-sum(KKK*Pmatg)

Isum_Estern[m]<-pi*(Isum1_Estern[m]/((b*T*F^2))-2*Isum2_Estern[m]/((b*T)^2*F^3)
                    +(Isum3a_Estern[m]*Isum3b_Estern[m])/((b*T)^3*F^4))
```

```
Isum_stern<-rep(0,N_BS)
Isum1_stern<-rep(0,N_BS)
Isum2_stern<-rep(0,N_BS)
Isum3a_stern<-rep(0,N_BS)
Isum3b_stern<-rep(0,N_BS)

Isum_sternMix<-rep(0,N_BS)
Isum1_sternMix<-rep(0,N_BS)
Isum2_sternMix<-rep(0,N_BS)
Isum3_sternMix<-rep(0,N_BS)
Isum4a_sternMix<-rep(0,N_BS)
Isum4b_sternMix<-rep(0,N_BS)

IsumBS<-double(N_BS)
IsumBS<-rep(0,N_BS)

  for(n in 1:N_BS)
  {
      k_BS<-rdunif(BS_g,F_in,-F_in)

        for(i in BS_u:BS_o)
        {
          Y_stern[(1+(i-1)*L):(i*L),n]<-Y[(1+(i-1)*L+k_BS[i-BS_u+1])
                                          :(i*L+k_BS[i-BS_u+1])]
          Z_stern[(1+(i-1)*L):(i*L),n]<-Z[(1+(i-1)*L+k_BS[i-BS_u+1])
                                          :(i*L+k_BS[i-BS_u+1])]
        }

  Yab_stern<-Y_stern[tvec,n]
  Zab_stern<-Z_stern[tvec,n]

  efYab_stern<-sapply(Yab_stern,function(x){sapply(Yab_stern,function(y){ef(x,y)})})
  efZab_stern<-sapply(Zab_stern,function(x){sapply(Zab_stern,function(y){ef(x,y)})})

  efPab_stern<-efYab_stern*efZab_stern

  Isum1_stern[n]<-sum(KK*efPab_stern)
  Isum3a_stern[n]<-sum(KK*efZab_stern)
  Isum3b_stern[n]<-sum(KK*efYab_stern)

  Pabmat_stern<-sapply(c(1:len),function(y){efZab_stern[,y]%*%t(efYab_stern[,y])})

  Isum2_stern[n]<-sum(KKK*Pabmat_stern)

  Isum_stern[n]<-pi*(Isum1_stern[n]/((b*T))-2*Isum2_stern[n]/((b*T)^2)
                   +(Isum3a_stern[n]*Isum3b_stern[n])/((b*T)^3))

  efYg_BS<-matrix(0,len,len)
  efZg_BS<-matrix(0,len,len)
  efPg_BS<-matrix(0,len,len)
  Pmatg_BS<-matrix(0,len^2,len)
  Pmatg2_BS<-matrix(0,len^2,len)

  for(r1 in -F_in:F_in)
  {
    Yr1_BS<-Y[(Kc1+r1+1):(Kc1+r1+len)]
    Zr1_BS<-Z[(Kc1+r1+1):(Kc1+r1+len)]

    efY_BS<-sapply(Yab_stern,function(x){sapply(Yr1_BS,function(y){ef(x,y)})})
    efZ_BS<-sapply(Zab_stern,function(x){sapply(Zr1_BS,function(y){ef(x,y)})})

    efP_BS<-efY_BS*efZ_BS

    efPg_BS<-efPg_BS+efP_BS
    efYg_BS<-efYg_BS+efY_BS
    efZg_BS<-efZg_BS+efZ_BS

    Pmat_BS<-sapply(c(1:len),function(y){efZ_BS[y,]%*%t(efY_BS[y,])})

    Pmatg_BS<-Pmatg_BS+Pmat_BS
```

```
225
226         for(r2 in -F_in:F_in)
227         {
228            Yr2_BS<-Y[(Kc1+r2+1):(Kc1+r2+len)]
229
230            efYr2_BS<-sapply(Yab_stern,function(x){sapply(Yr2_BS,function(y){ef(x,y)})})
231
232            Pmat2_BS<-sapply(c(1:len),function(y){efZ_BS[,y]%*%t(efYr2_BS[,y])})
233
234            Pmatg2_BS<-Pmatg2_BS+Pmat2_BS
235         }
236      }
237
238      Isum1_sternMix[n]<-sum(KK*efPg_BS)
239      Isum4a_sternMix[n]<-sum(KK*efZg_BS)
240      Isum4b_sternMix[n]<-sum(KK*efYg_BS)
241      Isum3_sternMix[n]<-sum(KKK*Pmatg_BS)
242      Isum2_sternMix[n]<-sum(KKK*Pmatg2_BS)
243
244      Isum_sternMix[n]<-pi*(Isum1_sternMix[n]/(T*b*F)-Isum2_sternMix[n]/((T*b*F)^2)
245                            -Isum3_sternMix[n]/((T*b)^2*F)
246                            +(Isum4a_sternMix[n]*Isum4b_sternMix[n])/((T*b)^3*F^2))
247
248      IsumBS[n]<-Isum_stern[n]-2*Isum_sternMix[n]+Isum_Estern[m]
249   }
250
251   qo_Isum[m]<-quantile(IsumBS,0.95)
252
253   if(Isum[m]<=qo_Isum[m])
254   {
255      Cov_Isum<-Cov_Isum+1
256   }
257 }
258
259 CovProz_Isum<-Cov_Isum/N
260
261 CovProz_Isum
```

Now we move on to the cases dealing with independence. Because the above-standing code is still valid except for the lines 80 to 90, we will merely present the adapted lines.

For innovations following the same distribution, we use the code fragment below:

```
epsY<-rstable(T,alpha,beta,gamma,mu)

Y[1]<-r(1/T)*rstable(1,alpha,beta,gamma,mu)+epsY[1]
Z[1]<-Y[1]

for(i in 2:T)
{
     Y[i]<-r(i/T)*Y[i-1]+epsY[i]
     Z[i]<-Y[i]
}
```

In case of log dependence as defined in Section 7.2, the code has to be altered in the following manner:

```
epsY<-rnorm(T,0,1)
epsZ<-log((epsY)^2)

epsY0<-rnorm(1,0,1)

Y[1]<-r(1/T)*epsY0+epsY[1]
Z[1]<-r(1/T)*log((epsY0)^2)+epsZ[1]
```

```
for(i in 2:T)
{
    Y[i]<-r(i/T)*Y[i-1]+epsY[i]
    Z[i]<-r(i/T)*Z[i-1]+epsZ[i]
}
```

Lastly, for the product dependence, whose definition can also be found in Section 7.2, we obtain the new lines as follows:

```
eps<-rnorm(T,0,1)

epsY<-rnorm(T,0,1)
epsZ<-epsY*eps

eps0<-rnorm(1,0,1)
epsY0<-rnorm(1,0,1)

Y[1]<-r(1/T)*epsY0+epsY[1]
Z[1]<-r(1/T)*epsY0*eps0+epsZ[1]

for(i in 2:T)
{
    Y[i]<-r(i/T)*Y[i-1]+epsY[i]
    Z[i]<-r(i/T)*Z[i-1]+epsZ[i]
}
```

With the last code modifications, we close this subsection and, therewith, the chapter as a whole.

Bibliography

Arcones, M.A. and Yu, B. (1994). Central limit theorems for empirical and U-processes of stationary mixing sequences. *Journal of Theoretical Probability* **7**, 47–71.

Bakirov, N.K., Rizzo, M.L., and Székely, G.J. (2006). A multivariate nonparametric test of independence. *Journal of Multivariate Analysis* **97**, 1742–1756.

Bronstein, I.N. and Semendjaew, K.A. (begründet), Grosche, G., Ziegler, V. and Ziegler, D. (weitergeführt), Zeidler, E. (Herausgeber) (2012). *Springer-Taschenbuch der Mathematik.* Springer Fachmedien Wiesbaden.

Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods.* Springer New York.

Cardinali, A. and Nason, G. (2010). Costationarity of locally stationary time series. *Journal of Time Series Econometrics* **2**, Article 1.

Dahlhaus, R. (1997). Fitting time series models to nonstationary processes. *The Annals of Statistics* **25**, 1–37.

Dahlhaus, R. (2012). Locally stationary processes. *Handbook of Statistics* **30**.

Dahlhaus, R., Richter, S. and W, W.B. (2018). Towards a general theory for non-linear locally stationary processes. *Bernoulli*, to appear.

Davis, R.A., Matsui, M. Mikosch, T. and Wan, P. (2018). Applications of distance correlation to time series. *Stochastic Processes and their Applications* **119**, 3699–3718.

Dehling, H., Durieu, O. and Volny, D. (2009). New techniques for empirical processes of dependent data. *Bernoulli* **24**, 3087–3116.

Dowla, A., Paparoditis, E. and Politis, D.N. (2003). Locally stationary processes and the Local Block Bootstrap. In: *Recent Advances and Trends in Nonparametric Statistics, (M.G. Akritas and D.N. Politis, Eds.)* Elsevier (North Holland), 437–444.

Dowla, A., Paparoditis, E. and Politis, D.N. (2013). Local block bootstrap inference for trending time series. *Metrika* **76**, 733–764.

Efron, B. (1979). Bootstrap methods: Another look at the jackknife. *The Annals of Statistics* **7**, 1–26.

Jentsch, C., Leucht, A., Meyer, M. and Beering, C. (2020). Empirical characteristic functions-based estimation and distance correlation for locally stationary processes. *Journal of Time Series Analysis* **41**, 110–133.

Kankainen, A. and Ushakov, N.G. (1998). A consistent modification of a test for independence based on the empirical characteristic function. *Journal of Mathematical Sciences* **89**, 1486–1494.

Kreiß, J.-P. and Neuhaus, G. (2006). *Einführung in die Zeitreihenanalyse*. Springer Berlin-Heidelberg.

Künsch, H.R. (1989). The jackknife and the bootstrap for general stationary observations. *The Annals of Statistics* **17**, 1217–1241.

Lahiri, S.N. (2003). *Resampling methods for dependent data*. Springer New York.

Liu, R.Y. and Singh, K. (1992).Moving blocks jackknife and bootstrap capture weak dependence. In: *R. Lepage and L. Billard, Eds., Exploring the Limits of Bootstrap* John Wiley, New York, 225–248.

Paparoditis, E. and Politis, D.N. (2002). Local block bootstrap. *Comptes rendus de l'Académie des sciences Paris, Série I* **335**, 959–962.

Priestley, M.B. (1965). Evolutionary spectra and non-stationary processes. *Journal of the Royal Statistical Society: Series B* **27**, 204–237.

Romano, J.P. and Wolf, M. (2000). A more general central limit theorem for m-dependent random variables with unbounded m. *Statistics & Probability Letters* **47**, 115–124.

Székely, G.J., Rizzo, M. L. and Bakirov, N.K. (2007). Measuring and testing dependence by correlation of distances. *The Annals of Statistics* **35**, 2769–2794.

van der Vaart, A.W. (1998). *Asymptotic Statistics*. Cambridge University Press.

van der Vaart, A.W. and Wellner, J.A. (2000). *Weak Convergence and Empirical Processes. With Applications to Statistics*. Springer New York.

Vogt, M. (2012). Nonparametric regression for locally stationary time series. *The Annals of Statistics* **40**, 2601–2633.

Wieczorek, B. (2016). Blockwise bootstrap of the estimated empirical process based on ψ-weakly dependent observations. *Statistical Inference for Stochastic Processes* **19**, 111–129.

Curriculum Vitae

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